L^2 techniques in Complex Geometry

by

Pietro Mesquita Piccione

Under the supervision of Sébastien Boucksom

Submitted to the IMJ-PRG
in partial fulfillment of the requirements for the degree of

Master in Mathematics

at the

SORBONNE UNIVERSITÉ

June 2022

Acknowledgments

I want to thank Sébastien Boucksom, for all the time we spent together. The Fondation Sciences Mathématiques de Paris for giving me the great opportunity of studying. And all my friends and family for all the support that they have given me.

Contents

Chapter 1

Introduction

This Master thesis is the result of some months of work under the supervision of Sébastien Boucksom, to whom I am very grateful.

This thesis is divided into three Chapters:

The first chapter is devoted to the introduction of some basics of Kahler Geometry; here we establish notations, and we give some results that will be employed at a later stage.

In the second chapter we discuss the notion of line bundles over a complex manifold, and we introduce the basic differential geometric notions required for the statement of the main results.

In the last chapter, we introduce the concept of plurisubharmonic function that will be an essential tool for employing measure theory and functional analysis techniques in Complex Geometry, which is the main core of the thesis. Using these techniques we prove a strong Cohomlogy vanishing theorem (Theorem 3.25) first, and then the celebrated Kodaira Embedding Theorem.

The Kodaira Embedding Theorem, is an important theorem in Complex Geometry/Complex Algebraic Geometry. It provides a link between the differential-geometric concept of *positive* line bundle and the algebro-geometric notion of *ampleness* for a line bundle.

The Kodaira Embedding Theorem also gives an answer to a natural question that arises in Complex Geometry: *When is a compact complex manifold, X, projective?*, that is, when does there exist an embedding of X into a complex projective space?

In order to illustrate the beauty and power of this theorem, let's assume the Kodaira Embedding Theorem and let us prove a simple corollary which gives an answer to the previous question.

Theorem 1.1. *If X is a compact complex manifold that admits a Kahler metric in $H^2(X,\mathbb{Z})$, then X is projective.*

Proof. If ω is such a metric, then by the holomorphic exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \mathcal{O} \xrightarrow{\ \exp\ } \mathcal{O}^* \longrightarrow 0 \tag{1.1}$$

we have the cohomology long exact sequence

$$\cdots \longrightarrow H^1(X,\mathcal{O}^*) \xrightarrow{\ s\ } H^2(X,\mathbb{Z}) \xrightarrow{\ r\ } H^2(X,\mathcal{O}) \cong H^{0,2}(X,\mathbb{C}) \longrightarrow \cdots \tag{1.2}$$

One can prove that the morphism r is exactly the the degree morphism, and therefore since ω is Kahler, in particular $1,1$, we have that $r(\omega)=0$. Hence $\omega \in \operatorname{Im}(s)$, which implies that there exists L a holomorphic line bundle over X such that $s(L)=c_1(L)=\omega$.

We conclude that L is a positive line bundle and X is projective by the Kodaira Embedding theorem. $\square$

1.1 Basic Kähler Geometry

Let $\mathbb{C}$ be the field of *complex numbers* $z=x+iy$. And $\mathbb{C}^n$ denote the coordinate space of the n-tuples of complex numbers $z=(z^1,\dots,z^n)$. If $U\subset\mathbb{C}^k$ is open, we say that a mapping $f\colon U\to\mathbb{C}^n$ is *smooth* if $\frac{\partial^{I,J} f}{\partial x^I \partial y^J}$ exists and is continuous for every I and J.

In these notes $\mathcal{A}^k(X) = C^\infty(X,\bigwedge^k T^*X)$ will denote the smooth k-differential forms.

Let (X, J) be a complex manifold, and g a Riemannian metric on X. We say that g is *compatible with the complex structure* if g is J-invariant. If g is compatible with J, then we say that (X, J, g) is *Kähler manifold* when

$$\nabla J = 0 \tag{1.3}$$

where ∇ is the Levi-Civita connection on X.

Remark 1.2. *The Kähler condition* (1.3) *is equivalent to J commuting with parallel transport.*

Since in a real Riemannian manifold homoteties commute with parallel transport, it is natural to ask the same for the complex case.

Associated to a compatible metric, g, we have a differential form ω_g, defined by

$$\omega_g(X, Y) \doteq -g(X, JY) = g(JX, Y)$$

Theorem 1.3. *Given a complex manifold (X, J) endowed with a compatible Riemannian metric g, (X, J, g) is Kähler if, and only if, ω_g is closed*

Proof. A direct computation of $d\omega_g$ shows that the following identity holds:

$$d\omega_g(X_0, X_1, X_2) = g\big((\nabla_{X_0} J)X_1, X_2\big) + g\big((\nabla_{X_1} J)X_2, X_0\big) + g\big((\nabla_{X_2} J)X_0, X_1\big).$$

This computation is made easier assuming[1] that X_0, X_1, X_2, JX_1 and JX_2 commute. Note that from this equality we obtain easily that if J is parallel, i.e., if (X, g, J) is Kähler, then ω_g is closed. We then have:

$$\begin{aligned} g\big((\nabla_{X_0} J)X_1, X_2\big) &= g\big((\nabla_{X_0} J)X_1, X_2\big) - g\big(J(\nabla_{X_0} X_1), X_2\big) \\ &= g\big((\nabla_{X_0} J)X_1, X_2\big) + g\big((\nabla_{X_0} X_1), JX_2\big). \end{aligned}$$

[1] One can always assume this on a complex manifold.

Using Koszul's formula, we get:

$$\begin{aligned}2g\big((\nabla_{X_0}J)X_1,X_2\big) &= X_0\big(g(JX_1,X_2)\big)+(JX_1)\big(g(X_0,X_2)\big)-X_2\big(g(X_0,JX_1)\big)\\ &= X_0\big(\omega_g(X_1,X_2)\big)-(JX_1)\big(\omega_g(JX_2,X_0)\big)+X_2\big(\omega_g(X_0,X_1)\big),\end{aligned}$$

and

$$\begin{aligned}2g\big(\nabla_{X_0}JX_1,JX_2\big) &= X_0\big(g(X_1,JX_2)\big)+X_1\big(g(X_0,JX_2)\big)-(JX_2)\big(g(X_0,X_1)\big)\\ &= -X_0\big(\omega_g(JX_1,JX_2)\big)+X_1\big(\omega_g(X_2,X_0)\big)-(JX_2)\big(\omega_g(X_0,JX_1)\big).\end{aligned}$$

Combining all the above identities we finally obtain:

$$2g\big((\nabla_{X_0}J)X_1,X_2\big)=d\omega_g(X_0,X_1,X_2)-d\omega(X_0,JX_1,JX_2),$$

which proves that if ω_g is closed, then J is parallel, i.e., (X,g,J) is Kähler. $\square$

1.1.1 Some Riemannian Geometry

Let (M,g) be a Riemannian manifold of even dimension[2]. Observe that for all $p\in M$, g_p induces a scalar product on T_p^*M. Indeed, g_p can be seen as a liner isomorphism

$$\begin{aligned}\flat\colon T_pM &\to T_p^*M\\ v &\mapsto g_p(v,\cdot)\doteq v^\flat\end{aligned}$$

we define in T_pM^* the sacalr product given by the push-forward $\flat_*(g_p)$. We will denote this scalar product by g_p as well.

The mapping inverse to $\flat$ will be denoted by $\sharp$.

More generally, g_p induces a scalar product in $\Lambda^k T_p^*M$. We define

$$g(\beta_1\wedge\cdots\wedge\beta_k,\alpha_1\wedge\cdots\wedge\alpha_k)\doteq\det\big(g(\beta_i,\alpha_j)\big),$$

[2]The evenness of dimension is not really needed, but it will make some sign computations easier.

and by linearity extend to $\Lambda^k T_p^* M$. If $\xi_1, \dots, \xi_n \in T^*M$ are orthonormal, and letting $I = \{i_1 < \cdots < i_k\}$, $J = \{j_1 < \cdots < j_{n-k}\}$ we have that

$$g(\xi_{i_1} \wedge \dots \xi_{i_k}, \xi_{j_1} \wedge \cdots \wedge \xi_{j_k}) = \delta_I^J \tag{1.4}$$

We'll define an operator from $C^\infty(\bigwedge^k T^*M)$ e $C^\infty(\bigwedge^{n-k} T^*M)$: the *Hodge star operator*.

Let $\beta \in C^\infty(\bigwedge^k T^*M)$ define

$$\varphi_\beta \colon \bigwedge^{n-k} T_p^* M \longrightarrow \mathbb{R}$$

such that $\varphi_\beta(\alpha) = g(\beta \wedge \alpha, dV_g)$. By Riesz representation theorem there exists a unique element, $*\beta \in \bigwedge^{n-k} T_p^* M$, such that

$$g(\beta \wedge \alpha, dV_g) = \varphi_\beta(\alpha) = g(\alpha, *\beta)$$

.

Since $\beta \mapsto *\beta$ is a linear function from $\bigwedge^k T_p^* M$ to $\bigwedge^{n-k} T_p^* M$, it induces a C^∞-linear function $C^\infty(\bigwedge^k T^*M) \to C^\infty(\bigwedge^{n-k} T^*M)$

$$*\colon \beta \mapsto *\beta$$

Definition 1.4. *The operatir above*

$$*\colon C^\infty(\bigwedge^k T^*M) \to C^\infty(\bigwedge^{n-k} T^*M)$$

is the Hodge star operator.

Let $\xi_1, \dots, \xi_n \in T_p^* M$ be a positively oriented basis.

Lemma 1.5. *The Hodge star is given by*

$$*(\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}) = \xi_{j_1} \wedge \cdots \wedge \xi_{j_{n-k}}$$

with $\xi_{i_1} \wedge \dots \xi_{i_k} \wedge \xi_{j_1} \wedge \dots \wedge \xi_{j_{n-k}} = (dV_g)_p$

Proof. By (1.4) we have that

$$\begin{aligned}
(\xi_{i_1} \wedge \dots \wedge \xi_{i_k}) &= \sum_L g((\xi_{i_1} \wedge \dots \wedge \xi_{i_k}), \xi_{l_1} \wedge \dots \xi_{l_{n-k}})\xi_{l_1} \wedge \dots \wedge \xi_{l_{n-k}} \\
&= \sum_L g(\xi_{i_1} \wedge \dots \wedge \xi_{i_k} \wedge \xi_{l_1} \wedge \dots \wedge \xi_{l_{n-k}}, dV_g)\xi_{l_1} \wedge \dots \wedge \xi_{l_{n-k}} \\
&= g(\xi_{i_1} \wedge \dots \wedge \xi_{i_k} \wedge \xi_{m_1} \wedge \dots \wedge \xi_{m_{n-k}}, dV_g)\xi_{m_1} \wedge \dots \wedge \xi_{m_{n-k}} \\
&= g(\operatorname{sgn}(\sigma) dV_g, dV_g)\xi_{m_1} \wedge \dots \wedge \xi_{m_{n-k}} \\
&= \operatorname{sgn}(\sigma)\xi_{m_1} \wedge \dots \wedge \xi_{m_{n-k}} \\
&= \xi_{j_1} \wedge \dots \wedge \xi_{j_{n-k}}
\end{aligned}$$

Where L runs through the $n-k$-multi-indexes, $\{m_1, \dots, m_{n-k}\}$ are the indices $\{j_1, \dots, j_{n-k}\}$ ordered, and σ the permutation associated to this ordering. □

The above lemma allow us to compute $*$-operator easily. One first consequence of this formula is

$$** = (-1)^k \mathrm{Id}$$

Which we get by a simple computation.

Corollary 1.6. $g(\beta, \alpha)dV_g = \beta \wedge *\alpha$

Proof.

$$\begin{aligned}
g(\beta, \alpha) =& (-1)^k g(\beta, **\alpha) \\
=& (-1)^k g(*\alpha \wedge \omega, dV_g) \\
=& g(\beta \wedge *\alpha, dV_g) \\
\implies& g(\beta, \alpha)dV_g = \beta \wedge *\alpha
\end{aligned}$$

□

A metric in M induces a scalar product $\bigwedge^k T_p^*M$ for every $p \in M$. Moreover, with these scalar products we can define a global scalar product on the differential forms,

$C^\infty(\bigwedge^k T^*M)$, integrating over M. If $\beta, \alpha \in C^\infty(\bigwedge^k T^*M)$ we define

$$\langle \beta, \alpha \rangle \doteq \int_M g(\beta, \alpha) dV_g = \int_M \beta \wedge *\alpha$$

We say that $\langle -, - \rangle$ is the *Hodge scalar product*, and we denote by $\|-\|$ the associated norm.

Therefore, if $\beta \in C^\infty(\bigwedge^k T^*M)$ and $\eta \in C^\infty(\bigwedge^{k+1} T^*M)$ the following computation holds:

$$\begin{aligned}
\langle d\beta, \eta \rangle &= \int_M d\beta \wedge *\eta \\
&= \int_M d(\beta \wedge *\eta) - \int_M (-1)^k \beta \wedge d(*\eta) \\
&= 0 + (-1)^{k+1} \int_M \beta \wedge d(*\eta) \\
&= (-1)^{k+1}(-1)^k \int_M \beta \wedge * * d(*\eta) \\
&= -\int_M \beta \wedge *(*d(*\eta)) \\
&= -\langle \beta, *d(*\eta) \rangle
\end{aligned}$$

We define $d^* = \delta \doteq - * d*$, so we have that

$$\langle \alpha, d\beta \rangle = \langle d^*\alpha, \beta \rangle$$

1.1.2 Back To Kähler

In the last section everything was done to a Riemannian manifold. For a Kähler manifold, we will extend the Hodge star operator, $*$, to be $\mathbb{C}$-linear in $T^*X \otimes \mathbb{C} \doteq T^*_{\mathbb{C}}X$.

In this way, we will have that for $v \in T^*X = T^*X^{1,0}$

$$\overline{*v} = *\overline{v}$$

And analogous results to the ones presented will be true for the Kähler case.

For instance, we can also define $\partial^* \doteq -*\overline{\partial}*$ and $\overline{\partial}^* \doteq -*\partial*$, we have

$$\langle \alpha, \partial\, \beta \rangle = \langle \partial^*\, \alpha, \beta \rangle \quad \text{and} \quad \langle \alpha, \overline{\partial}\, \beta \rangle = \langle \overline{\partial}^*\, \alpha, \beta \rangle \tag{1.5}$$

The proof is just as in the d case, with one key observation that if $\alpha \in C^\infty(\bigwedge^{n-1} T^*X)$, where $\dim_{\mathbb{C}} X = n$, we have

$$\int_X \partial\, \alpha = 0 = \int_X \overline{\partial}\, \alpha \tag{1.6}$$

Indeed, if $\alpha = \alpha^1 + \alpha^2$, where $\alpha^1 \in C^\infty(\bigwedge^{n,n-1} T^*X)$ and $\alpha^2 \in C^\infty(\bigwedge^{n-1,n} T^*X)$, then

$$\int_X \partial\, \alpha = \int_X \partial\, \alpha^2 = \int_X d\alpha^2 = 0$$

Similarly

$$\int_X \overline{\partial}\, \alpha = \int_X \overline{\partial}\, \alpha^1 = \int_X d\alpha^1 = 0$$

1.1.3 Kähler Identities

Let (X, ω) be a Kähler manifold, ω induces a linear homeomorphism

$$L_\omega \colon \mathcal{A}^k(X) \to \mathcal{A}^{k+2}(X)$$
$$\alpha \mapsto \alpha \wedge \omega$$

And also $\Lambda_\omega \doteq (-1)^k * L_\omega *$

Just like above, one can prove that:

$$\langle u, \Lambda_\omega v \rangle = \langle L_\omega u, v \rangle \tag{1.7}$$

For $u \in \mathcal{A}^k(X)$ and $v \in \mathcal{A}^{k+2}(X)$.

Remark 1.7. *But even better, we can prove that in each point $p \in X$:*

$$\begin{aligned}
g_p(L_\omega \beta, \alpha) = g_p(\omega \wedge \beta, \alpha) &= \frac{\omega \wedge \beta \wedge *\alpha}{dV_\omega} \\
&= \frac{\beta \wedge \omega \wedge *\alpha}{dV_\omega} = \\
&= \frac{\beta \wedge (-1)^k * *\omega \wedge *\alpha}{dV_\omega} = \\
&= \frac{\beta \wedge *(-1)^k * L_\omega * \alpha}{dV_\omega} = \\
&= \frac{\beta \wedge *\Lambda_\omega \alpha}{dV_\omega} = \\
&= g_p(\beta, \Lambda_\omega \alpha)
\end{aligned}$$

Therefore the duality is pointwise.

Let $\mathcal{H}^{p,q}(X, \mathbb{C})$ denote the harmonic p, q forms in X

We conclude the chapter stating a classical theorem in Hodge Theory.

Theorem 1.8 (Hodge Theorem)**.** *Let X be a compact Kähler manifold, then:*

$$H^{p,q}(X, \mathbb{C}) = H^{p,q}_{\overline{\partial}}(X) = \mathcal{H}^{p,q}(X, \mathbb{C}) \tag{1.8}$$

And

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$$

Proof. See [7]. ☐

Corollary 1.9. $H^{p,q}(X, \mathbb{C}) = \overline{H^{q,p}(X, \mathbb{C})}$

Proof. This follows easily from two observations on the Hodge star operator: we have that $*$ maps harmonic forms to harmonic forms, and that $\overline{\alpha}^{p,q}$ is q, p form, where $\alpha^{p,q}$ is of type p, q. ☐

Chapter 2

Line Bundles

If X is a Complex manifold we say $\pi\colon E \to X$ is a *line bundle*, if E is a Complex manifold, and there exists a covering of X, $\mathcal{U} \subset \mathcal{P}(X)$, such that for every U in $\mathcal{U}$, we find a chart ϕ such that the diagram commutes

$$\begin{tikzcd}
U \times \mathbb{C} \arrow[dr, "p"'] & & \pi^{-1}(U) \arrow[ll, "\phi"'] \arrow[dl, "\pi"] \\
& U &
\end{tikzcd} \tag{2.1}$$

Where p is the first coordinate projection.

With the additional property that if $\pi^{-1}(U) \cap \pi^{-1}(V) \neq \emptyset$, and $\psi\colon \pi^{-1}(V) \to V \times \mathbb{C}$ another such chart, then

$$\phi \circ \psi^{-1}(x, v) = (x, L_x v)$$

for $L_x \in \mathbb{C}$.

Example 2.1. *If X is a n-dimensional Complex manifold, then*

- $K_X \doteq \bigwedge^n T^*X$
- $-K_X \doteq \bigwedge^n TX$

are line bundles over X. In particular if X is a Riemann Surface, TX and T^*X are line bundles over X.

We'll call K_X the canonical bundle*, and $-K_X$ the* anti-canonical bundle*, over X.*

In a holomorphic line a bundle there is a natural way to define the holomorphicity of sections.

In a trivialization, $\tau\colon E\big|_U \to U \times \mathbb{C}$, we define $\overline{\partial}_E$ by the equation:

$$\overline{\partial}_E(fe) \doteq \overline{\partial} f \tag{2.2}$$

Where $e_x = \tau^{-1}(x,1)$, and $f \in C^\infty(X)$, since the transitions functions are holomorphic, we have that this operator is well defined. Indeed, if $g\overline{e} = e$ then

$$\overline{\partial}_E(fg\overline{e}) = \overline{\partial}(fg)\overline{e} = (\overline{\partial} f)g\overline{e} = (\overline{\partial} f)e$$

Where g is holomorphic and nowhere vanishing.

It's clear that:

$$\overline{\partial}_E^2 = 0$$

We say that a section $s \in C^\infty(X, E)$ is holomorphic if $\overline{\partial}_E\, s = 0$

2.1 Hermitian Line Bundles

In a line bundle $E \to X$, we can add a metric structure.

Definition 2.2. *Let $h \in C^\infty(X, E^* \otimes E^*)$ be a hermitian tensor, we say that h is a* metric *if for every $v \in TX \setminus \{0_p : p \in X\}$*

$$h(v, v) > 0$$

We say that (E, h) is an hermitian line bundle*.*

Remark 2.3. *In a trivialization, $\tau\colon E\big|_U \to U \times \mathbb{C}$, a metric in a line bundle is given by a function*

$$h(\xi, \xi) = e^{-2\phi(x)}|\tau(\xi)|^2$$

for $x \in U$ and $\xi \in E_x$.

In a holomorphic hermitian line bundle there is an natural connection, the complex analogue of the Levi-Civita connection.

Proposition 2.4. *Let (E, h) be an holomorphic hermitian line bundle over X. Then there is a unique connection ∇ such that*

- $\nabla^{0,1} = \overline{\partial}_E$
- $d(h(s,t)) = h(\nabla s, t) + h(s, \nabla t)$

We will call this connection, the *Chern connection* of (E, h), and will denote $\partial_E \doteq \nabla^{1,0}$, in such a way that the Chern connection will be the sum $\partial_E + \overline{\partial}_E$.

Let $\Theta(E)$ be curvature tensor associated to the metric in E. Locally we have

$$\Theta(E) = \sum c_{j,k} dz^j \wedge d\overline{z}^k$$

Let's consider the associated tensor:

$$\Theta^0(E)(\xi \otimes e) \doteq \sum c_{j,k}\, \xi^j\, \overline{\xi}^k$$

Definition 2.5. *We say that E is* positive *if $\Theta^0(E)(\xi \otimes e) > 0$ for all $\xi \neq 0$.*

In a trivialization like in 2.3, we have that $\Theta(E)$ is given by $2\,\partial\,\overline{\partial}\,\phi$. Up ahead we will study the class of functions φ such that $i\,\partial\,\overline{\partial}\,\varphi > 0$ is positive.

2.1.1 Kähler Identities (part 2)

Let X be an n-dimensional complex manifold, and consider the $\mathbb{Z}$-graded algebra $M^\bullet \doteq C^\infty(X, \wedge^\bullet T^* X)$; we say that an endomorphism $L \in \mathrm{End}(M^\bullet)$, is of *pure degree* $k \in \mathbb{Z}$, if

$$L\left[C^\infty(X, \wedge^a T^* X)\right] \subseteq C^\infty(X, \wedge^{a+k} T^* X)$$

for every $a \in \{0, 1, \ldots, n\}$

For endomorphisms of pure degree we define:

Definition 2.6. *Let $A, B \in \mathrm{End}(M^\bullet)$ endomorphisms of pure degree a and b respectively, we define the* graded Lie Bracket of A and B, *denoted by $[A, B]$, as the endomorphism*

$$[A, B] \doteq AB + (-1)^{ab} BA \tag{2.3}$$

Clearly the operators that we defined previously, such as L_ω, ∂ and $\overline{\partial}$, are endormorphisms of pure degree. In fact their degrees are respectively $2, 1$ and 1. Another important point is that if A is an endemorphism of pure degree then A^* is also of pure degree, with the opposite degree of A.

The last point we would like to mention is that there exists a Jacobi identity for the graded Lie Bracket, and it states that for A, B, C of degrees a, b, c respectively the following equation holds:

$$(-1)^{ca}[A, [B, C]] + (-1)^{ab}[B, [C, A]] + (-1)^{cb}[C, [A, B]] = 0$$

Theorem 2.7 (Kähler Identities for Line Bundles). *Let (X, ω) be a Kähler manifold, E an hermitian holomorphic vector bundle*[1]. *Then:*

$$[\overline{\partial}_E^*, L_\omega] = i\,\partial_E \tag{2.4}$$

$$[\partial_E^*, L_\omega] = -i\,\overline{\partial}_E \tag{2.5}$$

$$[\Lambda_\omega, \overline{\partial}_E] = -i\,\partial_E^* \tag{2.6}$$

$$[\Lambda_\omega, \partial_E] = i\,\overline{\partial}_E^* \tag{2.7}$$

Proof. This comes directly from the Kahler Identities in Hodge theory. □

Theorem 2.8 (Bochner-Nakano Identity).

$$\Delta_{\overline{\partial}} = \Delta_\partial + [i\Theta(E), \Lambda_\omega]$$

Proof. We have that

$$\Delta_{\overline{\partial}_E} = [\overline{\partial}_E, \overline{\partial}_E^*] = -i[\overline{\partial}_E, [\Lambda_\omega, \partial_E]]$$

[1]Vector bundles over a manifold are defined similarly to line bundles, but with fibers that are vector spaces of arbitrary dimension

where the last equality is given by (2.7). And analogously $\Delta_{\partial_E} = -i[\partial_E, [\overline{\partial}_E, \Lambda_\omega]]$

The Jacobi Identity for graded algebras gives us

$$[\overline{\partial}_E, [\Lambda_\omega, \partial_E]] = [\Lambda, [\partial_E, \overline{\partial}_E]] + [\partial_E, [\overline{\partial}_E, \Lambda_\omega]]$$

Now, since $\Theta(E)$ is a $1,1$ form we have that $\Theta(E) = [\partial_E, \overline{\partial}_E]$, which is the $1,1$ part of $d^2_\nabla = \Theta(E)$, we have that

$$\begin{aligned}\Delta_{\overline{\partial}_E} = -i[\overline{\partial}_E, [\Lambda_\omega, \partial_E]] = -i[\Lambda_\omega, \Theta(E)] + \Delta_{\partial_E} = \\ = \Delta_{\partial_E} + [i\Theta(E), \Lambda_\omega]\end{aligned}$$

$\square$

For X compact and $u \in C^\infty(X, \bigwedge^{p,q} T^*X \otimes E)$, we have the following:

$$\begin{aligned}\langle \Delta_\partial u, u\rangle &= \langle \partial\,\partial^*\, u + \partial^*\,\partial\, u, u\rangle && (2.8)\\ &= \langle \partial^*\, u, \partial^* u\rangle + \langle \partial\, u, \partial\, u\rangle && (2.9)\\ &= \|\partial^*\, u\|^2 + \|\partial\, u\|^2 && (2.10)\end{aligned}$$

In particular $\langle \Delta_\partial u, u\rangle \geq 0$.

Now, Theorem 2.8 implies that if u is $\overline{\partial}$-harmonic, i.e. $\Delta_{\overline{\partial}} u = 0$, then

$$\int_X \langle [i\Theta(E), \Lambda_\omega]u, u\rangle \, dV_\omega \leq 0 \tag{2.11}$$

Compute $[i\Theta(E), \Lambda_\omega]u$ is not trivial, but less trivial is to give a sign to it. In order to simplify the situation let's suppose that $p = n$, that is consider forms of the type $\bigwedge^{n,q} T^*X \otimes E$. Then we will have that:

$$\langle [i\Theta(E), \Lambda_\omega]u, u\rangle = \sum c_{j,k}\, u^j_S\, \overline{u^k_S} \tag{2.12}$$

Where $|S| = q$.

2.2 Positivity

Let's strip the differential geometry of Kähler manifold in one given point, to study only the Linear algebra of a complex hermitian vector space. The results will then that will be applied to each tangent space of each point of the manifold.

2.2.1 Linear Algebra

Let (V, J) be a complex vector space of $\dim_{\mathbb{C}} V = n$, let h be a hermitian scalar product on V. Given an orthonormal basis of V, one gets an orthornomal basis of V^*, $\xi_1, \ldots, \xi_n$, and define

$$\omega = \sum_j \xi_j \wedge \overline{\xi}_j$$

Lemma 2.9. *Let $\theta \in \bigwedge^{1,1} V^*$ be a real positive form. Then for every $q \geq 1$ we have*

1. *$[\theta, \omega^*]$, seen as a map $\bigwedge^{n,q} V^* \to \bigwedge^{n,q} V^*$, is positive definite, moreover if we have that $\theta \geq \epsilon\omega$, then*

$$[\theta, \omega^*] \geq q\epsilon \mathrm{Id} \colon \bigwedge^{n,q} V^* \to \bigwedge^{n,q} V^* \tag{2.13}$$

2. *$\langle [\theta, \omega^*]^{-1} u, u \rangle dV_\omega$ is decreasing w.r.t. ω and θ*

Proof. 1) Observe that, acting in $\bigwedge^{n,q} V^*$, $[\theta, \omega^*] = \theta\omega^*$ by the bidegree.

Let $\lambda_1 \leq \cdots \lambda_n$ be the eigenvalues of θ, observe that this makes sense since θ is real symmetric(since it is positive). Let $(\xi_j)_j$ be an h-orthonormal basis such that $\theta = i \sum_j \lambda_j \xi_j \wedge \overline{\xi}_j$. One can pick the orthonormal basis in the definition of ω to coincide with this basis. If we denote $\Omega \doteq \xi_1 \wedge \cdots \wedge \xi_n \in \bigwedge^{n,0} V^*$, we have that

$(\Omega \wedge \overline{\xi}_J)_{|J|=q}$ is a basis of $\bigwedge^{n,q} V^*$, and for $q = 1$ the following calculation holds:

$$
\begin{aligned}
[\theta, \omega^*](\Omega \wedge \overline{\xi}_j) &= \theta\omega^*(\Omega \wedge \overline{\xi}_j) = \\
&= (-1)^{n+1}\theta * \omega * (\xi_1 \wedge \cdots \wedge \xi_n \wedge \overline{\xi}_j) = \\
&= (-1)^{n+1}\theta * \omega \wedge (-1)^j(\overline{\xi}_1 \wedge \cdots \wedge \widehat{\xi_j} \cdots \wedge \overline{\xi}_n) = \\
&= (-1)^{n+1+j}\theta * (\sum_k \xi_k \wedge \overline{\xi}_k) \wedge (\overline{\xi}_1 \wedge \cdots \wedge \widehat{\overline{\xi}_j} \cdots \wedge \overline{\xi}_n) = \\
&= (-1)^{n+1+j}\theta * (\xi_j \wedge \overline{\xi}_j) \wedge (\overline{\xi}_1 \wedge \cdots \wedge \widehat{\overline{\xi}_j} \cdots \wedge \overline{\xi}_n) = \\
&= (-1)^{n+1+j}\theta * (-1)^{n-j+1}(\overline{\xi}_1 \wedge \cdots \wedge \overline{\xi}_j \cdots \wedge \overline{\xi}_n \wedge \xi_j) = \\
&= \sum_k \lambda_k \xi_k \wedge \overline{\xi}_k \wedge *(\overline{\xi}_1 \wedge \cdots \wedge \overline{\xi}_n \wedge \xi_j) \\
&= \sum_k \lambda_k \xi_k \wedge \overline{\xi}_k \wedge (-1)^{j+n}(\xi_1 \wedge \cdots \wedge \widehat{\xi_j} \cdots \wedge \xi_n)) \\
&= (-1)^{j+n}\lambda_j \xi_j \wedge \overline{\xi}_j \wedge (\xi_1 \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \xi_n) \\
&= (-1)^{j+n+n-j}\lambda_j \Omega \wedge \overline{\xi}_j = \lambda_j \Omega \wedge \overline{\xi}_j
\end{aligned}
$$

Similarly for $q \geq 1$ one has

$$
[\theta, \omega^*](\Omega \wedge \overline{\xi}_J) = \theta\omega^*(\Omega \wedge \overline{\xi}_J) = (\sum_{j \in J} \lambda_j)\Omega \wedge \overline{\xi}_J \tag{2.14}
$$

Then 2.13 follows easily.

For 2) see [2, Prop. 5.2, pag. 20] □

2.2.2 Curvature

Back to hermitian holomorphic line bundles:

Lemma 2.10. *There are local charts/local triviliaztions of X/E such that:*

$$
\omega = \sum_k \xi_k \wedge \overline{\xi}_k \quad / \quad h(e_i, e_j) = \delta_{i,j} + O(|z|^2)
$$

With the above lemma, we can use the results of Section 2.2.1, applied, in local trivializations, to differential forms. θ will be $\Theta(E)$, ω the Kähler metric (by the last lemma this definitions makes sense), by the remark 1.7 we have that $\omega^* = \Lambda_\omega$.

Observe that (2.12) follows.

If E is positive, then by 2.9 we have $[i\Theta(E), \Lambda_\omega]u \geq 0$, but not only that

$$\langle [i\Theta(E), \Lambda_\omega]u, u\rangle = 0 \implies u = 0$$

. In particular, if E is positive and u is $\overline{\partial}$-harmonic by 2.11 we have that $u = 0$.

Which by the Hodge Theorem implies that $H^{n,q}(X, E) = 0$

Theorem 2.11 (Nakano Vanishing Theorem)**.** *Let X be a compact complex manifold, E a* positive *line bundle, then*

$$H^{n,q}(X, E) = H^q(X, K_X \otimes E) = 0$$

Remark 2.12. *We can apply the Hodge Theorem even though we don't assume that X is Kähler. One can do that because, if X assumes a positive vector bundle, then*

$$Tr(i\Theta(E)) = i\sum c_{j,k}\, dz^j \wedge d\overline{z}^k$$

is a Kähler metric.

Similarly to (2.10) we have that

$$\langle \Delta_{\overline{\partial}} u, u\rangle = \|\overline{\partial}^*\, u\|^2 + \|\overline{\partial}\, u\|^2$$

Which together with Theorem2.8 gives us

Proposition 2.13. $\|\overline{\partial}^* u\|^2 + \|\overline{\partial}\, u\|^2 \geq \int_X \langle [i\Theta(E), \Lambda_\omega]u, u\rangle\, dV_\omega$

Chapter 3

L^2 Theory

General mantra:

In differential geometry is customary to consider only smooth objects, since those are the natural maps to study in a smooth setting. But for some analytical results it is useful to consider completions of spaces of functions, and use the powerful tools of functional analysis to prove theorems.

3.1 Currents

Distributions are defined as dual objects of compacly supported smooth functions on a manifold. Likewise, *currents* in a differentiable manifold are dual objects of compactly supported smooth differential forms. The topology of a manifold M^n defines a a natural topology on the space of smooth compactly supported differential forms of order $n-q$, that will be denoted by $C_c^\infty(X, \bigwedge^{n-q} T^*X)$. Let us discuss this in some detail.

Let K_n be an exhaustion of X by compact sets, and let

$$C_c^\infty(K) \subset C_c^\infty(X, \bigwedge^{n-q} T^*X)$$

be the set of smooth $n-q$ differential forms with support contained in K. We will now define the topology of $C_c^\infty(K_n)$, and by a inductive process this will be enough to determine te topology of $C_c^\infty(X, \bigwedge^{n-q} T^*X)$, for more details see [3] and [11].

In a compact set K, we can always find a finite cover of K, $(K_\alpha)_{\alpha=1,\dots,F}$, by compact sets, such that each $K_\alpha \subseteq U_\alpha$ is contained in a coordinate neighborhood U_α.

Now for every α we may define a family of semi-norms[1] on $C_c^\infty(K)$:

$$\|\beta\|_{\alpha,\ell} \doteq \max_{|a|\leq \ell} \sup_{K_\alpha} |D^a b_{I,\alpha}| \tag{3.1}$$

Where $\beta\big|_{K_\alpha} = \sum_{|I|=n-q} b_{I,\alpha} dx_\alpha^I$ in local coordiates, D is the differential, and a a multi-index.

We therefore have a countable family of semi-norms on $C_c^\infty(K)$:

$$\big\{\|\bullet\|_{\alpha,\ell} : \ell \in \mathbb{N}, \alpha = 1, \dots, F\big\}$$

Which we will denote by $\big\{\|\bullet\|_j\big\}_{j\in\mathbb{N}}$, define now the distance function:

$$d(\eta, \sigma) \doteq \sum_{j\geq 1} \frac{\|\eta - \sigma\|_j}{1 + \|\eta - \sigma\|_j} \frac{1}{2^j} \tag{3.2}$$

for $\eta, \sigma \in C_c^\infty(K)$. This defines a metric space structure on $C_c^\infty(K)$, and since K is compact is easy to see that $\big(C_c^\infty(K), d\big)$ is complete.

We clearly then have that if a sequence $(\eta_j)_j \in C_c^\infty(K)$ converges to η_∞, then for every α and for every $\ell \in \mathbb{N}$

$$\|\eta_j - \eta_\infty\|_{\alpha,\ell} \to 0$$

but not only that, we have that this convergence is uniform with regard to α and ℓ.

This means that taking $\beta_j^\alpha \doteq (\eta_j - \eta_\infty)\big|_{K_\alpha}$, when written in local coordinates $\beta_j^\alpha = \sum b_{j,I}^\alpha dx_\alpha^I$, is such that $\lim_j \sup_{K_\alpha} |D^a b_{j,I}^\alpha| = 0$, uniformely in α and in a.

Now we can take the strict locally convex inductive limit[2] of $\big(C_c^\infty(K_n), d_n\big)$, to obtain a topology on $C_c^\infty(X, \bigwedge^{n-q} T^*X)$. It is known that such topology is not metrizable when M is noncompact. We will not give here an explicit description of this

[1]Semi-norms are exactly like norms but they are not point-separing, that is an element might have semi-norm equal to zero but not be zero.

[2]See [3], or [11].

topology. However, we remark that that this inductive limit is such that a functional $f\colon C_c^\infty(X, \bigwedge^{n-q} T^*X) \to \mathbb{R}$ is continuous if and only if $f_n \doteq f\big|_{C_c^\infty(K_n)}$ is continuous as function defined in $\big(C_c^\infty(K_n), d_n\big)$, see for instance [6, Theorem 25, p. 10] Hence we have that f is continuous iff for every n and every sequence $x_j \in C_c^\infty(K_n)$ such that $x_j \to x$, we have

$$f(x_j) \to f(x)$$

.

If a sequence $(\eta_j)_j \in C_c^\infty(X, \bigwedge^{n-q} T^*X)$ is such that the support of every η_j is contained in a single fixed compact K of X, we say that $(\eta_j)_j$ is an *admissible sequence*.

Definition 3.1. *A* current of degree q *on* M*, or a* q-current*, is an element of the topological dual space to* $C_c^\infty(X, \bigwedge^{n-q} T^*X)$*.*

That is, we say that

$$T\colon C_c^\infty(X, \bigwedge^{n-q} T^*X) \to \mathbb{C}$$

a linear map is a q-current if for every admissible sequence $\alpha_k \in C_c^\infty(X, \bigwedge^{n-q} T^*X)$ *that converges to* 0*,*

$$T(\alpha_k) \to 0$$

In particular, distributions are currents of maximal degree.

Given current of degree q on M, and a smooth compactly supported differential form ω on M, we will denote by $\langle \alpha, \omega \rangle \in \mathbb{R}$ the value of α at ω.

The space of q-forms emnbeds naturally in the space of q currents. More generally, any p-form α whose coefficients are locally integrable functions can be seen as a current, by setting $\langle \alpha, \omega \rangle = \int_M \alpha \wedge \omega$.

Just as in the case of distributions, one can define a series of important operations on currents, like for instance the *restriction to open subsets*, or the *wedge product with a differential form.* It is also defined in a natural way the operation of pointwise multiplication by a smooth function, which makes the space of currents of a fixed order a $C^\infty(M)$-module.

Another important operation that can be extended to currents is that of differentiation. In particular, for a current α on a complex manifold one can define $\overline{\partial}\,\alpha$. The following regularity result holds:

Theorem 3.2. *If α is a $\overline{\partial}$-closed current, then α is in fact a smooth differential form.*

Proof. Since $\Delta = i\Lambda\partial\,\overline{\partial}$, it follows easily from the corresponding result for the regularity of harmonic p-forms, see for instance [7]. □

Definition 3.3. *A* $1,1$ *current T is said to be* positive *if for every strictly positive $n-1, n-1$ form $\alpha \in C_c^\infty(X, \bigwedge^{n-1,n-1} T^*X)$, $T(\alpha) \geq 0$.*

In local coordinates we have that $T = \sum \tau_{i,j} dz^i \wedge d\overline{z}^j$ is said to be positive *if*

$$\sum \tau_{i,j} \lambda^i \overline{\lambda}^j$$

is a positive measure[3] *for every $(\lambda^1, \cdots, \lambda^n) \in \mathbb{C}^n$*

Remark 3.4. *We can actually see the space of q-currents as the tensor product*

$$C^\infty(X, \bigwedge^q T^*X) \otimes_{C^\infty} \mathcal{D}(X)$$

Where $\mathcal{D}(X)$ denotes the space of distributions, dual to $C_c^\infty(X)$.

Seeing it this way, it is clear that all the natural notions on differential forms (e.g. type, wedge products, derivations,..) can be extended to currents.

3.2 Differential Operators

Let E and F be complex vector bundles over X.

Definition 3.5. *A* linear differential operator of order $\mathfrak{o}$ *is a $\mathbb{C}$-linear map*

$$P \colon C^\infty(X, E) \to C^\infty(X, F)$$

[3] We say that a measure is positive if it is positive on the positive functions

such that in a trivialization, P is of the form:

$$P = \sum_{|I| \leq \mathfrak{o}} a_I D^I$$

where $I = (i_1, \ldots, i_k)$ is a multi-index, $|I| = i_1 + \ldots + i_k$, $a_I \in \hom(E, F)$, and $D^I = (\frac{\partial}{\partial x^1})^{i_1} (\frac{\partial}{\partial x^2})^{i_2} \cdots (\frac{\partial}{\partial x^k})^{i_k}$.

We will deal mostly with differential operators of order $|\mathfrak{o}| \leq 1$. For these operators we define the *symbol* as a smooth section σ_P of $TX \otimes \hom(E, F)$ given by the formula:

$$P(fu) = fPu + (\sigma_P f)u \tag{3.3}$$

Where P is first order linear differential operator, σ_P it's symbol, $f \in C^\infty(X)$, and $u \in C^\infty(X, E)$.

Example 3.6. *If E is an hermitian holomorphic line bundle, $P \doteq \overline{\partial}_E$ is a linear differential operator of order $\mathfrak{o} = 1$.*

If E and F are hermitian, there is a natural unbounded operator (in the functional analytic sense) associated to a linear differential operator P. Indeed, we can associate to P the operator

$$T_P \colon D(T_P) \subset L^2(M, E) \to L^2(M, F) \tag{3.4}$$

$$f \mapsto Pf \tag{3.5}$$

Where Pf is calculated as a distribution, and

$$D(T_P) \doteq \left\{ f \in L^2(X, E) : Pf \in L^2(X, F) \right\}$$

It is clear that $C_c^\infty(X, E) \subset D(T_P)$, which implies that T is an (densely defined) unbounded operator.

Moreover, T_P is closed, indeed if $u_\nu \xrightarrow{L^2} u$ and $T_P u_\nu \xrightarrow{L^2} v$, then as distributions

$$Pu_\nu \to Pu$$

since for every $f \in C_c^\infty$

$$
\begin{aligned}
|\int_X (Pu_\nu - Pu)\overline{f}| = |\int_X (u_\nu - u)\tilde{P}f| &\leq \int_X |u_\nu - u||\tilde{P}f| \\
&= \int_K |u_\nu - u||\tilde{P}f| \\
&\leq C\mu(K)\|u_\nu - u\| \to 0
\end{aligned}
$$

where $\tilde{P}$ is given by the definition of the derivative as a distribution(integration by parts formula), K the support of f.

By a similar argument one may prove that L^2 convergence implies point-wise distribution convergence. Thus we have that $Pu(f) = v(f)$ as distributions, and this implies that $u \in D(T_P)$ and $T_P(u) = v$. We conclude that T_P is a closed densely defined operator.

3.2.1 Adjoint

There are two different notions for adjoints of differential operators: the one given by T_P and A.3, in the functional analytic L^2 sense, and a distribution theoretic one, which we will say the *formal adjoint*.

Definition 3.7. *If P is a differential operator in hermitian vector bundles, it exists a unique differential operator*

$$P^\star \colon C^\infty(X, F) \to C^\infty(X, E)$$

such that

$$\langle Pu, v\rangle = \langle u, P^\star v\rangle \tag{3.6}$$

For either u or v having compact support. We will say that $P^\star$ is the formal adjoint *of P.*

Proof. This is a simple argument using partitions of unity and integration by parts, that allows to conclude that

$$P^{\star}v(x) = \sum_{|I|\leq \mathfrak{o}} (-1)^{|I|} g(x)^{-1} D^I (g(x)\overline{a}_I^{\star} v(x) \tag{3.7}$$

where $g(x)$ is given by the formula $g dx^1 \cdots dx^n = dV_g$. □

We can go further an extend $P^{\star}$ to L^2 functions in the following sense: We define $D(P^{\star}) \doteq \{f \in L^2 : \exists h_f \in L^2$ such that $\langle Dg, f\rangle = \langle g, h_f\rangle$ for every $g \in C_c^{\infty}\}$, and we get an operator

$$P^{\star} \colon D(P^{\star}) \subseteq L^2 \to L^2$$

which we will still denote it by $P^{\star}$.

Example 3.8. *In Section 1.1, we defined the operators Λ_ω, d^*, $\overline{\partial}^*$, and ∂^*. It is clear that they are all indeed the formal adjoints of the differential operators L_ω, d, $\overline{\partial}$, and ∂.*

In Section 3.2 we concluded that associated to a differential operator P, there is an closed densely defined operator on L^2, T_P. By the general theory of unbounded operators in a Hilbert space (c.f. Theorem A.3), we have that there is an densely defined closed operator, T_P^*.

Definition 3.9. *We will say that T_P^* is the* Hilbert adjoint of P, *and we will denote it sometime as T_P^* or even P^*.*

Observe that these two definitions do not agree, indeed:

Example 3.10 (Formal $\neq$ Hilbert Adjoint)**.** *Let M a real Riemannnian manifold given by $]0,1[$ with usual (scalar) product.*

$$P = \frac{d}{dx} \colon L^2(]0,1[) \to L^2(]0,1[)$$

First observe that by the Fundamental Theorem of Calculus for Measure Theory *we have $D(P) \subseteq C^0([0,1])$.*

By definition $D(P^*) = \{f \in L^2(]0,1[) : \exists h_f \in L^2$ *such that* $\langle Pg, f\rangle = \langle g, h\rangle$, $\forall g \in D(P)\}$, *that is* $f \in D(P^*)$ *if and only if we can find* $h_f \in L^2$ *such that*

$$\int_0^1 (\frac{d}{dx}g)f = \int_0^1 gh_f$$

for all $g \in D(P)$. *Since* $C_c^\infty \subseteq D(P)$, *we have that* $D(P^*) \subseteq D(P)$, *and* $h_f = P^*f = -\frac{d}{dx}f$. *But from the integration by parts formula we have that for* $g \in C^\infty$ *and* $f \in D(P^*)$:

$$\begin{aligned}\int_0^1 (\frac{d}{dx}g)f &= f(1)g(1) - f(0)g(0) - \int_0^1 g\frac{d}{dx}f = \\ &= f(1)g(1) - f(0)g(0) + \int_0^1 gP^*f\end{aligned}$$

Therefore we have that $f(1) = 0 = f(0)$ *(this makes sense since* $D(P) \subseteq C^0([0,1])$*). In fact this shows that* $D(P^*) = \{f \in D(P) : f(1) = 0 = f(0)\}$

But it is clear that when we consider the formal adjoint, $P^\star$, *we get that the domain of* $P^\star$ *coincides with* $D(P)$, *since* $f \in D(P^\star)$ *if and only if it exists* $h_f \in L^2$ *such that*

$$\int_0^1 f(x)(Pg)(x)dx = \int_0^1 h_f(x)g(x)dx$$

for every $g \in C_c^\infty$, *and by the integration by parts formula*

$$\begin{aligned}\int_0^1 (\frac{d}{dx}g)f &= f(1)g(1) - f(0)g(0) - \int_0^1 g\frac{d}{dx}f = \\ &= 0 + \int_0^1 gP^*f\end{aligned}$$

we have that $h_f(x) = -\frac{d}{dx}f(x)$, *and we don't have the restriction as we did in the* P^* *case.*

Therefore $D(P^*) \subsetneq D(P^\star)$.

3.3 L^2 Theory on Complete Riemannian Manifolds

As seen before, in general the notions of adjoint, namely the formal adjoint and the Hilbert adjoint, do not agree. But if we add the hypothesis of X admiting a complete riemannian metric, the issue disappears for the first order operators we are interested in ∇_E and $\overline{\partial}_E$, and we have

Theorem 3.11. *Let (X, g) be* complete*, then for every $u \in L^2(M, E)$:*

- *We can find $u_n \in C_c^\infty(X, E)$ such that $u_n \xrightarrow{L^2} u$, with the following property: for every $P\colon C^\infty(X, E) \to C^\infty(X, F)$ first order linear differential operator, with bounded symbol σ_P, $Pu_n \xrightarrow{L^2} Pu$, if $Pu \in L^2(X, F)$.*
- $P^\star = P^*$

Proof. For the first: By the Friedrich's Lemma in the theory of differential operators we can restrict ourselves to the case that $u \in C^\infty$. Then, since the manifold is complete, we can find a sequence of smooth cut-off functions $\chi_n\colon X \to \mathbb{R}$ such that $d\chi_n$ is bounded, and if $K_n \doteq \chi_n^{-1}(1)$ then it satisfies

$$\cup K_n = X$$

.

Define $u_n \doteq \chi_n u$, clearly $u_n \in C_c^\infty$, and we have that:

$$P(u_n) = P(\chi_n u) = \sigma_P(\chi_n)u + \chi_n P(u)$$

By the dominated convergence theorem we have that $\chi_n P(u) \to P(u)$ in L^2. Since σ_P is bounded we have that $|\sigma_P(f)| \leq C|df|$, where C is a constant, and for every f. Since $d\chi_n$ is bounded, we have $|\sigma_P(\chi_n)|$ is bounded.

Since $\sigma_P(\chi_n)(t)$ is eventually 0 for every $t \in X$, we have that, by the dominated convergence theorem, $\sigma_P(\chi_n)u \to 0$ in L^2. We therefore have that

$$P(u_n) \xrightarrow{L^2} P(u)$$

For the second point: Both adjoints agree where defined, in particular they agree on smooth compactly supported sections, using the first point and passing through a limit gives the result. □

Example 3.12. *For $P = \overline{\partial}_E$, $\overline{\partial}_E^*$, ∇_E, and ∇_E^* the result applies, i.e. σ_P is bounded.*

With this we also get a improved version of Proposition 2.13.

Corollary 3.13. *If g is complete, then if*

$$u \in L^2(X, \bigwedge^{n,q} T^*X \otimes L) \cap D(T_{\overline{\partial}}) \cap D(T_{\overline{\partial}}^*)$$

we have

$$\|\overline{\partial}^* u\|^2 + \|\overline{\partial}\, u\|^2 \geq \int_X \langle [i\Theta(E), \Lambda_\omega] u, u\rangle \, dV_\omega \tag{3.8}$$

Proof. By Theorem 3.27 we can find a sequence of functions $u_n \in C_c^\infty(X)$ such that $u_n \to u$ and $Pu_n \to Pu$, for P either $\overline{\partial}$ or $\overline{\partial}^*$, we then have that

$$\int_X \langle [i\Theta(E), \Lambda_\omega] u_n, u_n\rangle \, dV_\omega \leq \|\overline{\partial}\, u_n\|^2 + \|\overline{\partial}^*\, u_n\|^2 \to \|\overline{\partial}\, u\|^2 + \|\overline{\partial}^*\, u\|^2$$

We then can apply the Fatou's Lemma for a subsequence of u_n, which we will still denote by u_n that converges almost everywhere to u, and have that

$$\int_X \langle [i\Theta(E), \Lambda_\omega] u, u\rangle \, dV_\omega \leq \liminf \int_X \langle [i\Theta(E), \Lambda_\omega] u_n, u_n\rangle \, dV_\omega \leq \|\overline{\partial}\, u\|^2 + \|\overline{\partial}^*\, u\|^2$$

□

3.4 Main Results

Theorem 3.14 (Complicated Smooth Version)**.** *Let (X, ω) be a complete Kähler manifold, and E a positive line bundle. If $v \in L^2(X, \bigwedge^{n,q} T^*X \otimes E)$, is $\overline{\partial}_E$-closed, i.e., $\overline{\partial}_E\, v = 0$ with*

$$\int_X \langle [i\Theta(E), \Lambda_\omega]^{-1} v, v\rangle \, dV_\omega < \infty$$

then

$$\overline{\partial}\, u = v$$

for some $u \in L^2(X, \bigwedge^{n,q-1} T^*X \otimes E)$ *and it satisfies the following norm inequality:*

$$\|u\|^2 \leq \int_X \langle [i\Theta(E), \Lambda_\omega]^{-1} v, v\rangle \, dV_\omega$$

Proof. Let A denote the operator $[i\Theta(E), \Lambda_\omega]$.

Since $\ker T_{\overline{\partial}_E}$ is closed, we have that

$$L^2(X, \bigwedge^{n,q} T^*X \otimes E) = \ker T_{\overline{\partial}_E} \oplus (\ker T_{\overline{\partial}_E})^{\perp} \tag{3.9}$$

Let $f \in D(T_{\overline{\partial}^*_E})$, f decomposes as $f_1 + f_2$, with $f_1 \in \ker T_{\overline{\partial}_E}$ and $f_2 \in (\ker T_{\overline{\partial}_E})^{\perp}$. And we have

$$\begin{aligned}
|\langle v, f\rangle|^2 = |\langle v, f_1\rangle|^2 = |\langle A(A^{-1}v), f_1\rangle|^2 \leq \\
&\leq \Big(\int \langle A(A^{-1}v), A^{-1}v\rangle dV_\omega\Big)\Big(\int \langle Af_1, f_1\rangle\Big) \\
&= \Big(\int \langle v, A^{-1}v\rangle dV_\omega\Big)\Big(\int \langle Af_1, f_1\rangle\Big)
\end{aligned}$$

Where the inequality is the above formula is is the Cauchy–Schwarz inequality, using that $A \geq 0$.

Observe that, by Theorem A.3, we have

$$(\ker T_{\overline{\partial}_E})^{\perp} = \overline{\mathrm{Im} T^*_{\overline{\partial}_E}} \subset \ker T_{\overline{\partial}^*_E}$$

which implies that $\overline{\partial}^*_E\, f_2 = 0$. Since $f \in D(T_{\overline{\partial}^*_E})$, it follows that $f_1 \in D(T_{\overline{\partial}^*_E})$ as well, therefore $f_1 \in D(T_{\overline{\partial}^*_E}) \cap D(T_{\overline{\partial}_E})$, by Corollary 3.13 we have:

$$\begin{aligned}
\int_X \langle [i\Theta(E), \Lambda_\omega] f_1, f_1\rangle \, dV_\omega &\leq \|\overline{\partial}^* f_1\|^2 + \|\overline{\partial}\, f_1\|^2 \\
&= \|\overline{\partial}^* f_1\|^2 = \|\overline{\partial}^* f\|^2
\end{aligned}$$

Therefore we get

$$|\langle v,f\rangle|^2 \leq \Big(\int \langle v, A^{-1}v\rangle dV_\omega\Big)\|\overline{\partial}^* f\|^2 \tag{3.10}$$

For every $f \in D(T_{\overline{\partial}_E^*})$.

This implies that we have a well defined continuous function:

$$\ell\colon T_{\overline{\partial}_E^*}(D(T_{\overline{\partial}_E^*})) \subseteq L^2 \to \mathbb{C}$$
$$\overline{\partial}^* f \mapsto \langle f, v\rangle$$

with $\|\ell\| \leq C \doteq \big(\int \langle A^{-1}v, v\rangle\big)^{\frac{1}{2}}$. We can extend ℓ to L^2 setting it to be zero in the orthogonal complement of $T_{\overline{\partial}_E^*}(D(T_{\overline{\partial}_E^*}))$, and by the Riesz Representation theorem it exists

$$u \in L^2(X, \bigwedge^{n,q-1} T^*X \otimes E)$$

such that

$$\langle \overline{\partial}^* f, u\rangle = \langle f, v\rangle \tag{3.11}$$

For every $f \in D(T_{\overline{\partial}_E^*})$, which is dense in L^2, we then conclude that $\overline{\partial}\, u = v$. □

Theorem 3.15 (Reasonable Smooth Version)**.** *Let (X,ω) a complete Kähler manifold, and E a holomorphic hermitian line bundle such that*

$$\Theta(E) \geq \epsilon\omega$$

for some $\epsilon > 0$. Then for every $\overline{\partial}_E$-closed form, $v \in L^2(X, \bigwedge^{n,q} T^*X \otimes E)$, there exists $u \in L^2(X, \bigwedge^{n,q-1} T^*X \otimes E)$ with

$$\overline{\partial}_E\, u = v$$

such that

$$\|u\|^2 \leq \frac{1}{q\epsilon}\|v\|^2$$

Proof. This is a direct application of Lemma 2.9 and Theorem 3.14. □

Remark 3.16. *The same results are true, if instead of supposing that (X,ω) is com-*

plete, we instead suppose that X admits a complete Kähler metric, ω'.

Proof. Indeed, let $\omega_\epsilon \doteq \omega + \epsilon\omega'$, it is not hard to verify that ω_ϵ is complete. In general if $\omega_1 \geq \omega_2$ Kähler (Riemannian) metrics, with ω_2 complete, ω_1 will be complete as well.

Then, since $\omega_\epsilon \geq \omega$, by Lemma 2.9 we have that

$$\langle [i\Theta(E), \Lambda_{\omega_\epsilon}]^{-1} v, v\rangle dV_{\omega_\epsilon} \leq \langle [i\Theta(E), \Lambda_{\omega}]^{-1} v, v\rangle dV_{\omega}$$

Therefore if v is such that $\int_X \langle [i\Theta(E), \Lambda_\omega]^{-1} v, v\rangle dV_\omega < \infty$, then so will

$$\int_X \langle [i\Theta(E), \Lambda_{\omega_\epsilon}]^{-1} v, v\rangle dV_{\omega_\epsilon} < \infty$$

we can then apply the theorem and get $u_\epsilon \in L^2_{\omega_\epsilon}(X, T^*X \otimes E)$ that solves

$$\overline{\partial}\, u_\epsilon = v$$

with the estimate

$$\|u_\epsilon\|^2_{\omega_\epsilon} \leq \int_X \langle [i\Theta(E), \Lambda_{\omega_\epsilon}]^{-1} v, v\rangle \, dV_{\omega_\epsilon} \leq \int_X \langle [i\Theta(E), \Lambda_{\omega}]^{-1} v, v\rangle \, dV_{\omega}$$

Since $\omega_\epsilon \geq \omega$ then $dV_{\omega_\epsilon} \geq dV_\omega$ and $|s|_{\omega_\epsilon} \geq |s|_\omega$ as well. Therefore

$$u_\epsilon \in L^2_\omega(X, \bigwedge^{n,q-1} T^*X \otimes E)$$

with bounded norm, by the Banach–Alaoglu theorem, we have that there exists a weakly convergent sequence $u_{\epsilon_k} \overset{L^2}{\rightharpoonup} u$, with $u \in L^2_\omega(X, T^*X \otimes E)$, such that

$$\|u\|^2 \leq \int_X \langle [i\Theta(E), \Lambda_{\omega}]^{-1} v, v\rangle \, dV_{\omega}$$

And clearly we have that in the sense of distributions

$$\overline{\partial}\, u = v$$

$\square$

3.4.1 Plurisubharmonic Functions and Convexity

Let $\Omega \subseteq \mathbb{C}^n$ be an open set, we define now a class o functions that plays an important role in the theory.

Definition 3.17. *Let $u\colon \Omega \to [-\infty, \infty[$ be an upper-semicontinous function, we say that u is* plurisubharmonic, *or* psh, *if for every $a \in \Omega$, and $\xi \in \mathbb{C}^n$ such that the 2-disk $\{a + z\xi : z \in \mathbb{D}\} \subset \Omega$, we have*

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + \xi e^{i\theta}) d\theta \tag{3.12}$$

A clear first consequence is that the maximum of two psh functions is psh

Lemma 3.18. *Let $u_1, u_2\colon \Omega \to [-\infty, \infty[$ be psh functions, then $f \doteq \max\{u_1, u_2\}$ is psh.*

Proof. Let $a \in \Omega$ and $\xi \in \mathbb{C}^n$ such that $\{a + z\xi : z \in \mathbb{D}\} \subset \Omega$. Then clearly

$$\int_0^{2\pi} u_j(a + \xi e^{i\theta}) d\theta \leq \int_0^{2\pi} f(a + \xi e^{i\theta}) d\theta, \quad \text{for } j = 1, 2$$

hence

$$u_j(a) \leq \int_0^{2\pi} u_j(a + \xi e^{i\theta}) d\theta \leq \int_0^{2\pi} f(a + \xi e^{i\theta}) d\theta$$

since $f(a) \in \{u_1(a), u_2(a)\}$ the result follows. $\square$

There is a useful characterization of L^1_{loc} psh functions.

Proposition 3.19. *$u \in L^1_{loc}(\Omega)$ is plurisubharmonic if, and only if, $i\partial\overline{\partial}u \geq 0$, in the sense of currents*[4].

Proof. For simplicity, let us assume that u is $C^2(\Omega)$. In this case, the result is obtained from the following computation.

[4]See Section 3.1.

Let $F(t) \doteq \frac{1}{2\pi}\int_0^{2\pi} u(a+t\xi e^{i\theta})d\theta$, then

$$\frac{1}{2\pi}\int_0^{2\pi} u(a+\xi e^{i\theta})d\theta - u(a) = F(1) - F(0) = \int_0^1 F'(t)dt =$$
$$= \frac{1}{2\pi}\int_0^1 \Big(\int_0^{2\pi} \langle \nabla u(a+t\xi e^{1\theta}), \xi e^{i\theta}\rangle d\theta\Big)dt = \star$$

Taking $G(s) \doteq \int_0^{2\pi} \langle \nabla u(a+s\xi e^{1\theta}), \xi e^{i\theta}\rangle d\theta$, we have that $\int_0^t G'(s)ds = G(t) - G(0) = G(t)$, and

$$\begin{aligned}
\star &= \frac{1}{2\pi}\int_0^1 G(t)dt = \frac{1}{2\pi}\int_0^1 \Big(\int_0^t G'(s)ds\Big)dt \\
&= \frac{1}{2\pi}\int_0^1 \Big(\int_0^t \Big(\int_0^{2\pi} \langle \operatorname{Hess} u(a+s\xi e^{i\theta})\xi e^{i\theta}, \xi e^{i\theta}\rangle d\theta\Big)ds\Big)dt = \\
&= \frac{1}{2\pi}\int_0^1 \Big(\int_0^t \frac{1}{s}\Big(\int_0^{2\pi} s\langle \operatorname{Hess} u(a+s\xi e^{i\theta})\xi, \xi\rangle d\theta\Big)ds\Big)dt = \\
&= \frac{1}{2\pi}\int_0^1 dt\Big(\int_{|z|\le t} \frac{\operatorname{Hess} u(a+z\xi)}{|z|} d\lambda(z)\Big)
\end{aligned}$$

Where λ is the Lebesgue measure in $\mathbb{C}$.

Therefore, since $\star \ge 0$ so is $\frac{1}{2\pi}\int_0^1 dt\big(\int_{|z|\le t} \frac{\operatorname{Hess} u(a+z\xi)}{|z|} d\lambda(z)\big) \ge 0$, and we have that $\operatorname{Hess} u$ is positive semi-definite. □

We say that φ is *strictly psh* if $i\,\partial\,\overline{\partial}\,\varphi > 0$.

Using the characterization of psh functions given in Proposition 3.19, one can extend the concept of plurisubharmonicity for functions defined on complex manifolds.

Definition 3.20. *A complex manifold X is said to be* weakly pseudo-convex *if it admits a smooth exhaustion function*[5] *that is psh. The manifold X is* strongly pseudo-convex *if it admits a smooth exhaustive function that is strictly psh.*

Clearly, every compact manifold is weakly pseudo-convex, as one sees considering constant functions.

We will now give the statement of a standard result of the theory, whose proof can be found in [2].

[5] We say that a continuous function, ψ, is exhaustive if $\psi^{-1}(]-\infty, c]) \Subset X$

Proposition 3.21. *Every weakly pseudo-convex Kähler manifold admits a* complete *Kähler metric.* □

A *good exhaustion* of X is an exhaustion by weakly pseudo-convex open sets $\Omega_j \Subset X$, with compact closure in X.

Definition 3.22. *Given a function $\varphi \in L^1_{loc}$ and a $(1,1)$ closed real form θ, we say that φ is θ-*regularizable *if, φ is θ-psh, that is, such that $\theta + i\,\partial\,\overline{\partial}\,\varphi \geq 0$, and if there exists a sequence $\varphi_j \in C^\infty(X)$ such that:*

a) $\varphi_j(p) \to \varphi(p)$ for every $p \in X$;

b) for every $\Omega \Subset X$ open set with compact closure, there exists $\overline{j}(\Omega)$ such that the sequence $(\varphi_j\big|_\Omega)_j$ is decreasing and strictly θ-psh for $j \geq \overline{j}(\Omega)$.

Theorem 3.23 (Even More Complicated Singular Version)**.** *Let (X, ω) be a Kähler manifold, with good exhaustion. E an holomorphic line bundle over X, with the singular metric $\tilde{h} = e^{-\varphi}h$ such that*

$$i\Theta(E, \tilde{h}) = i\Theta(E, h) + i\,\partial\,\overline{\partial}\,\varphi \geq \eta > 0$$

where φ is a $(i\Theta(E, h) - \eta)$-regularizable function, and η a positive $(1,1)$-form.

If $v \in L^2(X, \bigwedge^{n,q} T^*X \otimes E)$, a $\overline{\partial}_E$-closed form, with

$$\int_X \langle [\eta, \Lambda_\omega]^{-1} v, v\rangle\, e^{-2\varphi} dV_\omega < \infty$$

then

$$\overline{\partial}\,u = v$$

for some $u \in L^2(X, \bigwedge^{n,q-1} T^*X \otimes E)$ that satisfies the following norm inequality:

$$\|u\|^2_{\tilde{h}} \leq \int_X \langle [\eta, \Lambda_\omega]^{-1} v, v\rangle\, e^{-2\varphi} dV_\omega$$

Proof. Since X has a good exhaustion and φ is regularizable we have: $\{\Omega_j \Subset X\}_j$, an exhaustion of X by weakly pseudoconvex open sets, for each j a sequence $(\varphi^j_k)_k$ of smooth functions on Ω_j that converge decreasingly to $\varphi\big|_{\Omega_j}$, since $i\Theta(E,h)+i\,\partial\,\overline{\partial}\,\varphi\geq\eta$ we can assume that $i\Theta(E,h)+i\,\partial\,\overline{\partial}\,\varphi_k^j\geq\eta$ as well.

Since $\varphi_k^j\geq\varphi$, we have that

$$\int_{\Omega_j}\langle[\eta,\Lambda_\omega]^{-1}v,v\rangle\,e^{-2\varphi_k^j}dV_\omega\leq\int_X\langle[\eta,\Lambda_\omega]^{-1}v,v\rangle\,e^{-2\varphi}dV_\omega\doteq C$$

By Lemma 2.9, we can apply the smooth theorem to $(E,e^{-\varphi_k^j}h)$ obtaining that there exists $u_k^j\in L^2_{j,k}(\Omega_j)$ such that

$$\overline{\partial}_E\,u_k^j=v$$

and

$$\|u_k^j\|^2_{j,k}\leq\int_{\Omega_j}\langle[\eta,\Lambda_\omega]^{-1}v,v\rangle\,e^{-2\varphi_k^j}dV_\omega\leq C$$

Since $-\varphi_\ell^j\leq-\varphi_k^j$ for $k\geq\ell$, we have that $\|u_k^j\|^2_{j,\ell}\leq\|u_k^j\|^2_{j,k}\leq C$, therefore $(u_k^j)_{k\geq\ell}\in L^2_{j,\ell}$ is a bounded sequence. This implies that there exists a subsequence that weakly converges in $L^2_{j,\ell}$, by a Cantor diagonal argument we can find a subsequence of $(u_k^j)_k$, which we will still denote by the same symbol, such that

$$u_k^j\overset{L^2_{j,\ell}}{\rightharpoonup}u^j$$

for all $\ell\geq1$.

So far we have that for each Ω_j we found a u^j such that $\overline{\partial}_E\,u^j=v$ and $\|u^j\|^2_{j,\ell}\leq C$ for every ℓ. This implies that actually $\|u^j\|^2_j\leq C$ by the monotone convergence theorem we have that

$$\int_{\Omega_j}|u^j|^2e^{-\varphi}dV_\omega=\int_{\Omega_j}|u^j|^2\lim_\ell e^{-\varphi_\ell^j}dV_\omega=\lim_\ell\int_{\Omega_j}|u^j|^2e^{-\varphi_\ell^j}dV_\omega\leq C$$

We then have that for every compact $K\subset X$ there exists $m_K\geq1$ such that for $j\geq m_K$ $K\subset\Omega_j$, and then $(u^j\big|_K)_{j\geq m_K}\in L^2(K)$ becomes a bounded sequence, which therefore has a weakly convergent subsequence $u^j\overset{L^2(K)}{\rightharpoonup}u\in L^2(K)$ with $\|u\|^2\leq C$

and s.t.

$$\overline{\partial}_E u = v$$

The theorem then follows. $\square$

Theorem 3.24 (Reasonable Singular Version)**.** *Let (X, ω) be a Kähler manifold, with good exhaustion. E an holomorphic line bundle over X, with the singular metric $\tilde{h} = e^{-\varphi} h$ such that*

$$i\Theta(E, \tilde{h}) = i\Theta(E, h) + i\,\partial\,\overline{\partial}\,\varphi \geq \epsilon\omega \quad \big(i\Theta(E \otimes K_X^*) = i\Theta(E, h) + i\,\partial\,\overline{\partial}\,\varphi + Ric(\omega) \geq \epsilon\omega\big)$$

with φ a $i\Theta(E, h) - \epsilon\omega$-regularizable ($i\Theta(E, h) - \epsilon\omega + Ric(\omega)$-regularizable) function, and $\epsilon > 0$. Then for every $\overline{\partial}_E$-closed form,

$$v \in L^2(X, \bigwedge^{n,q} T^*X \otimes E) \quad \big(v \in L^2(X, \bigwedge^{0,q} T^*X \otimes E)\big),$$

there exists

$$u \in L^2(X, \bigwedge^{n,q-1} T^*X \otimes E) \quad \big(u \in L^2(X, \bigwedge^{0,q-1} T^*X \otimes E)\big)$$

with

$$\overline{\partial}_E u = v$$

such that

$$\|u\|_{\tilde{h}}^2 \leq \frac{1}{q\epsilon} \|v\|_{\tilde{h}}^2$$

Proof. For the n, q case we have that by Lemma 2.9, since $i\Theta(E, \tilde{h}) = i\Theta(E, h) + i\,\partial\,\overline{\partial}\,\varphi \geq \epsilon\omega$, we have that $[i\Theta(E, \tilde{h}), \Lambda_\omega]^{-1} \leq \frac{1}{q\epsilon}\mathrm{Id}$, and by Theorem 3.23, the result holds.

For $0, q$ case it is enough to observe that a $0, q$ form with values in E can be seen as a n, q form with values in $E - K_X$ by contraction, and in these case the line bundle has curvature

$$\Theta(E) + Ric(\omega)$$

And we restricted ourselves to the previous case. $\square$

Define $J(\varphi) \doteq \{f \in \mathcal{O}_X : f \in L^2_\varphi\}$ as the ideal of the sheaf of holomorphic functions of X. It consists of of germs of holomorphic functions such that $|f|^2 e^{-2\varphi}$ is integrable.

Theorem 3.25 (Vanishing Cohomology)**.** *Let (X, ω) be a weak pseudo convex Kähler manifold, E a holomorphic line bundle, with metric h such that there exists*

$$\varphi\colon X \to [-\infty, \infty[\text{ a } L^1_{loc} \text{ function}$$
$$\epsilon > 0$$

such that φ is $(i\Theta(E,h) - \epsilon\omega)$-regularizable.

Then:

$$H^q\big(X, \mathcal{O}(K_X + E) \otimes J(\varphi)\big) = 0$$

for $q \geq 1$.

Proof. Let $\mathcal{L}^q$ be the sheaf of germs of sections of $\bigwedge^{n,q} T^*X \otimes E$, u, such that both u and $\overline{\partial}_E u$ belong to $L^2(e^{-\varphi}h)$. we then have that:

$$\mathcal{L}^0 \xrightarrow{\overline{\partial}_E} \mathcal{L}^1 \xrightarrow{\overline{\partial}_E} \cdots \xrightarrow{\overline{\partial}_E} \mathcal{L}^n \longrightarrow 0 \tag{3.13}$$

is a complex. This will be a resolution of $\mathcal{O}(K_X + E) \otimes J(\varphi)$.

Indeed we have a natural homomorphism $i\colon \mathcal{O}(K_X + E) \otimes J(\varphi) \to \mathcal{L}^0$, and by the regularity theory of currents, the image of i is equal to the kernel of $\overline{\partial}_E$. To check the exactness in the rest of the sequence, it is enough to apply Theorem 3.24 to our funtion φ in small open neighborhoods around each point. Therefore

$$0 \longrightarrow \mathcal{O}(K_X + E) \otimes J(\varphi) \xrightarrow{i} \mathcal{L}^0 \xrightarrow{\overline{\partial}_E} \mathcal{L}^1 \xrightarrow{\overline{\partial}_E} \cdots \tag{3.14}$$

is exact, and since $\mathcal{L}^q$ is a C^∞ sheaf module over X, we have that $(\mathcal{L}^\star, \overline{\partial}_E)$ is a acyclic resolution. This means that we can compute the cohomology of $\mathcal{O}(K_X + E) \otimes J(\varphi)$ using the resolution. In fact,

$$H^q\big(X, \mathcal{O}(K_X + E) \otimes J(\varphi)\big) = H^q(\mathcal{L}^\star(X), \overline{\partial}_E)$$

where $H^q(\mathcal{L}^\star(X), \overline{\partial}_E)$ is the cohomology of the global sections of $\mathcal{L}$.

Just like we did for the sheaf exactness, we would like to use Theorem 3.24 to conclude that the global sections have vanishing cohomology. But, in Theorem 3.24 it is assumed that the sections are L^2 in X, while here the global sections of $\mathcal{L}$ need only to be *locally* L^2. In order to circumvent this problem, we use the weak pseudo-convexity, and we apply Theorem 3.24 with a different metric, $e^{-\chi\circ\psi}h$, where ψ is a smooth psh exhaustion function of X and $\chi\colon \mathbb{R} \to \mathbb{R}$ a convex smooth function that grows arbitrarily fast at infinity. So we get that the sections are L^2 w.r.t. this new metric, and therefore Theorem 3.24 gives that for every $\overline{\partial}_E$-closed $v \in \mathcal{L}^q(X)$ is $\overline{\partial}_E$ exact in $L^2(e^{-\chi\circ\psi}h)$, and in particular in $\mathcal{L}$. And the result follows. $\square$

Let's now extend the notion of psh function with the introduction of the introduce a new class of functions called quasi-psh. Roughly speaking, a quasi-psh function is a function which is smooth, except at some point where the singularity is that of a psh function.

Definition 3.26 (Quasi-psh functions)**.** *We say that a upper-semicontinous function* $u\colon X \to [-\infty, \infty[$ *is* quasi-psh *if locally it is of the form* $f + \varphi$ *where* f *is psh, and* φ *is smooth.*

The following theorem show us that quasi-psh functions are well approximated by smooth functions.

Theorem 3.27 (Approximation of quasi-psh functions)**.** *Let* X *be a complex manifold,* $\varphi\colon X \to [-\infty, +\infty]$ *a continuous quasi-psh function, and* $\theta > 0$ *a closed, positive* $(1,1)$ *form such that*

$$\theta + i\,\partial\overline{\partial}\,\varphi \geq 0$$

Then φ *is* θ*-regularizable.*

Proof. Let $\psi_j \doteq \max\{-j, \varphi\}$, since θ is closed, locally we may find $f \in C^\infty(X)$ such that $i\,\partial\overline{\partial}\,f = \theta$, and it follows that $f - j$ and $f + \varphi$ are psh, therefore $\max\{f - j, f + \varphi\} = f + \psi_j$ is psh, hence $\theta + i\,\partial\overline{\partial}\,\psi_j \geq 0$.

Consider an exhaustion of X by compact sets $K_j \subset \overset{\circ}{K}_{j+1}$, we will construct a sequence $\varphi_j \in C^\infty(X)$ such that for every j φ_j we have:

$$\varphi_j(p) \to \varphi(p),\ \forall p \in X \tag{3.15}$$

$$\psi_j\big|_{K_{j+1}} < \varphi_j\big|_{K_{j+1}} \tag{3.16}$$

$$\varphi_{j+1}\big|_{K_{j+1}} \leq \varphi_j\big|_{K_{j+1}} \tag{3.17}$$

$$\theta + i\,\partial\overline{\partial}\,\varphi_j\big|_{K_{j+1}} > 0 \tag{3.18}$$

Once done that $(\varphi_j)_j$ will be the sequence needed in the definition of θ-regularizable, that is φ_j will be eventually decreasing and strictly θ-psh on compact sets.

Indeed, since $(K_j)_j$ is exhaustive on X we have that, given a compact set K, there exists $j_0 \in \mathbb{N}$ such that $K \subseteq K_{j_0}$, and by (3.17) and (3.18), part b) of the definition 3.22 is satisfied.

Let's then construct such a sequence. We will do it by recurrence. Let φ_0 be any smooth function such that $\varphi_0 > \psi_0$. Now suppose we have constructed $\varphi_0, \varphi_1, \ldots, \varphi_{j-1}$. Since ψ_j is a continuous *quasi-psh* and θ-psh, we have that, by Richberg's theorem, there exists a *decreasing* sequence $(\psi_j^k)_k$ such that:

$$\left(\psi_j^k(p)\right)_k \to \psi_j(p) \text{ for every } p \in X$$

and on K_j:

$$\theta + i\,\partial\overline{\partial}\,\psi_j^k > -\epsilon_k\theta$$

for some positive sequence $\epsilon_k \to 0$. For more details on this see [2][Theorem 3.4]

Now, set $\delta_k \doteq \frac{1}{1+\epsilon_k} < 1$, and define

$$\varphi_j^k \doteq \delta_k\psi_j^k + (1-\delta_k)(1 + \sup_{K_{j+1}} \psi_j)$$

If we restrict our function to K_{j+1} we will have:

$$\varphi_j^k \geq \delta_k \psi_j + (1-\delta_k)(1+\sup_{K_{j+1}} \psi_j) = \delta_k \psi_j + (1-\delta_k)\sup_{K_{j+1}} \psi_j + 1 - \delta_k >$$
$$> \delta_k \psi_j + (1-\delta_k)\sup_{K_{j+1}} \psi_j \geq \psi_j$$

Which implies that for every $k \in \mathbb{N}$:

$$\varphi_j^k\big|_{K_{j+1}} > \psi_j\big|_{K_{j+1}} \tag{3.19}$$

Again on K_j we have:

$$\begin{aligned}\theta + i\,\partial\overline{\partial}\,\psi_j^k > -\epsilon_k\theta \implies & (1+\epsilon_k)\theta + i\,\partial\overline{\partial}\,\psi_j^k > 0 \\ \implies & 0 < \theta + i\,\partial\overline{\partial}\,\delta_k\psi_j^k = \\ = \theta + i\,\partial\overline{\partial}[\delta_k\psi_j^k + (1-\delta_k)(1+\sup_{K_{j+1}}\psi_j)] &= \theta + i\,\partial\overline{\partial}\,\varphi_j^k\end{aligned}$$

obtaining

$$(\theta + i\,\partial\overline{\partial}\,\varphi_j^k)\big|_{K_j} > 0 \tag{3.20}$$

for every $k \in \mathbb{N}$.

By Dini's theorem we have that on K_j the pointwise limit of $(\psi_j^k)_k$, is actually an uniform limit. That is:

$$\psi_j^k\big|_{K_j} \underset{k\to\infty}{\overset{C^0}{\rightrightarrows}} \psi_j\big|_{K_j} \tag{3.21}$$

Now, take $k(j) \in \mathbb{N}$ large enough in order to the following conditions hold:

1. $\sup_{K_j}|\psi_j^k - \psi_j| \leq \frac{1}{j}$ for every $k \geq k(j)$
2. $(1-\delta_{k(j)})\sup_{K_j}|\psi_j^{k(j)}| \leq \frac{1}{j}$
3. $\psi_j^{k(j)}\Big|_{K_j} < \varphi_{j-1}\big|_{K_j}$
4. $|(1+\sup_{K_{j+1}}\psi_j)(1-\delta_{k(j)}| \leq \frac{1}{j}$

Let's check that there exists such a $k(j)$.

Indeed, by (3.21) we have that $\sup_{K_j}|\psi_j^k - \psi_j| \to 0$, and $\sup_{K_j}|\psi_j^{k(j)}|$ is bounded, therefore we can find such a $k(j)$.

For the third point, we have that $\psi_j \leq \psi_{j-1}$, and by construction $\psi_{j-1} < \varphi_{j-1}$, therefore it exists $\eta_j > 0$ such that

$$0 < \eta_j < \varphi_{j-1} - \psi_j = \varphi_{j-1} - \lim_k \psi_j^k$$

hence there is $k(j)$ such that for $k \geq k_0(j)$, $\varphi_{j-1} - \psi_j^k \geq \eta_j > 0$ on K_j.

Lastly, $|(1 + \sup_{K_{j+1}} \psi_j) \geq 1$, and since $\delta_k \to 1$ we can find such $k(j)$.

Now, set $\varphi_j \doteq \varphi_j^{k(j)}$, and it remains to prove that $\varphi_j(p) \to \varphi(p)$ for every p.

Let $p \in X$, take $j_0 \in \mathbb{N}$ such that $p \in K_{j_0}$, and we will have that for every $j \geq j_0$ it holds:

$$\begin{aligned}
|\varphi_j(p) - \varphi(p)| &\leq |\delta_{k(j)}\psi_j^{k(j)}(p) - \varphi(p)| + |(1 + \sup_{K_{j+1}} \psi_j)(1 - \delta_{k(j)}| \leq \\
&\leq |(1 - \delta_{k(j)})\psi_j^{k(j)}| + |\psi_j^{k(j)}(p) - \varphi(p)| + \frac{1}{j} \leq \\
&\leq \frac{1}{j} + |\psi_j^{k(j)}(p) - \psi_j(p)| + |\psi_j(p) - \varphi(p)| + \frac{1}{j} \leq \\
&\leq \frac{3}{j} + |\psi_j(p) - \varphi(p)| \xrightarrow[j \to \infty]{} 0
\end{aligned}$$

$\square$

Definition 3.28 (Log-like functions). *Let X be a compact complex manifold. Given $p, q \in X$, $p \neq q$, a* log-like function *(centered at p and q) is a function $\varphi_{p,q}\colon X \to [-\infty, \infty[$ such that*

- *$\varphi_{p,q} = n \log|z - p|$ in local coordinates around p*
- *$\varphi_{p,q} = n \log|z - q|$ in local coordinates around q*
- *$\varphi_{p,q} \in C^\infty(X \setminus \{p, q\})$*

Remark 3.29. *A log-like function $\varphi_{p,q}$ is quasi-psh (it is psh near p and q, and smooth elsewhere).*

Remark 3.30. *Given a log-like function $\varphi_{p,q}$ on X, we have that $J(\varphi) = m_{p,q}$, where $m_{p,q}$ is the ideal of holomorphic functions that vanish at p and q.*

Corollary 3.31. *Let (X, ω) be a compact Kahler manifold, E a holomorphic line bundle on X with metric h, $\varphi_{p,q}$ a log-like function such that:*

$$
\begin{aligned}
&i\Theta(E,h) + i\,\partial\overline{\partial}\,\varphi_{p,q} > 0, \quad \textit{and} \\
&i\Theta(E,h) > 0
\end{aligned}
$$

Then there exists a surjective mapping:

$$H^0(X, \mathcal{O}(K_X + E)) \twoheadrightarrow (K_X + E)_p \oplus (K_X + E)_q$$

Proof. Since X is compact there exists $\epsilon > 0$ such that both inequalities hold:

$$
\begin{aligned}
&i\Theta(E,h) + i\,\partial\overline{\partial}\,\varphi_{p,q} - \epsilon\omega > 0 \\
&\eta \doteq i\Theta(E,h) - \epsilon\omega > 0
\end{aligned}
$$

By Theorem 3.27, $\varphi_{p,q}$ is η-regularizable.

By Theorem 3.25, we have that $H^1(X, \mathcal{O}(K_X + E) \otimes m_{p,q}) = 0$.

Then observe that the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(K_X + E) \otimes J(\varphi_{p,q}) \xrightarrow{\;i\;} \mathcal{O}(K_X + E) \xrightarrow{\;\pi\;} \mathcal{O}(K_X + E) \otimes \mathcal{O}_X / J(\varphi_{p,q}) \longrightarrow 0 \tag{3.22}$$

Gives a long exact sequence on the cohomology

$$H^0\big(\mathcal{O}(K_X + E)\big) \longrightarrow H^0\big(X, \mathcal{O}(K_X + E) \otimes \mathcal{O}_X / J(\varphi_{p,q})\big) \longrightarrow H^1\big(X, \mathcal{O}(K_X + E) \otimes J(\varphi_{p,q})\big) \tag{3.23}$$

Since the last term is 0, by exactness we have the surjective morphism

$$H^0(X, \mathcal{O}(K_X + E)) \twoheadrightarrow H^0\big(X, \mathcal{O}(K_X + E) \otimes \mathcal{O}_X / J(\varphi_{p,q})\big)$$

Now, since $J(\varphi_{p,q}) = m_{p,q}$, we have that the stalk of $\mathcal{O}(K_X + E) \otimes \mathcal{O}_X/J(\varphi_{p,q})$ in each point $x \notin \{p, q\}$ is equal to zero, since $(\mathcal{O}_X/m_{p,q})_x = 0$. Therefore for each small local extension, s, of $s_p \in \mathcal{O}(K_X + E)_p \otimes \big(\mathcal{O}_X/J(\varphi_{p,q})\big)_p$ satisfies

$$s_x = 0$$

for each $x \in U(p) \setminus \{p\}$ a neighborhoodof p. This gives us that for every $s_1 \in \mathcal{O}(K_X + E)_p \otimes \big(\mathcal{O}_X/J(\varphi_{p,q})\big)_p$ and $s_2 \in \mathcal{O}(K_X + E)_q \otimes \big(\mathcal{O}_X/J(\varphi_{p,q})\big)_q$ we can define $s \in H^0\big(X, \mathcal{O}(K_X + E) \otimes \mathcal{O}_X/J(\varphi_{p,q})\big)$ by:

$$\begin{aligned} &s_x = 0, \ \ x \neq p, q \\ &s_p = s_1 \\ &s_q = s_2 \end{aligned}$$

Hence, we have found a surjection:

$$\begin{aligned} &H^0\big(X, \mathcal{O}(K_X + E) \otimes \mathcal{O}_X/J(\varphi_{p,q})\big) \\ &\qquad \twoheadrightarrow \mathcal{O}(K_X + E)_p \otimes (\mathcal{O}_X/J(\varphi_{p,q}))_p \oplus \mathcal{O}(K_X + E)_q \otimes (\mathcal{O}_X/J(\varphi_{p,q}))_q \end{aligned}$$

The result follows from:

$$\begin{aligned} (K_X + E)_p \oplus (K_X + E)_q &= \mathcal{O}(K_X + E)_p \oplus \mathcal{O}(K_X + E)_q = \\ &= \mathcal{O}(K_X + E)_p \otimes \mathbb{C} \oplus \mathcal{O}(K_X + E)_q \otimes \mathbb{C} = \\ &= \mathcal{O}(K_X + E)_p \otimes (\mathcal{O}_X/m_{p,q})_p \oplus \mathcal{O}(K_X + E)_q \otimes (\mathcal{O}_X/m_{p,q})_q = \\ &= \mathcal{O}(K_X + E)_p \otimes (\mathcal{O}_X/J(\varphi))_p \oplus \mathcal{O}(K_X + E)_q \otimes (\mathcal{O}_X/J(\varphi))_q \end{aligned}$$

□

For a function φ such that $\varphi(z) = 2n \log|z - p|$ in local coordinates around p and smooth elsewhere, we have that $J(\varphi) = m_p^2$, and then

$$\begin{aligned} \mathcal{O}_X/J(\varphi))_p &\to \mathbb{C} \oplus T_p^* X \\ [f] &\mapsto (f(p), d_p f) \end{aligned}$$

is an isomorphism, which implies

$$\mathcal{O}(K_X + E)_p \otimes (\mathcal{O}_X/J(\varphi))_p = (K_X + E)_p \otimes (\mathbb{C} \oplus T_p^* X)$$

Using exactly the same proof as above we conclude that there exists a surjective mapping

$$H^0(X, \mathcal{O}(K_X + E)) \twoheadrightarrow (K_X + E)_p \otimes (\mathbb{C} \oplus T_p^* X) \tag{3.24}$$

Remark 3.32. *In the proof of the Corollary 3.31, we could have avoided the use of Theorem 3.25. To illustrate the point, let us prove that, for all fixed $q \in X$, the evaluation map:*

$$H^0(X, \mathcal{O}(K_X + E)) \twoheadrightarrow (K_X + E)_q$$

is surjective. To this aim, choose an arbitrary $s(q) \in (E + K_X)_q$, and let $s \colon U(q) \to E + K_X$ be a local holomorphic section around $q \in X$, with $s_q = s(q) \in (E + K_X)_q$. Let $\chi \colon X \to \mathbb{R}$ a smooth function, such that $\chi\big|_{X \setminus V_2(q)} \equiv 0$, and $\chi\big|_{V_1(q)} \equiv 1$, where $V_1(q) \subset V_2(q)$ are neighborhoods of q.

Define $\tilde{s} \doteq \chi s$, as a global smooth section of $K_X + E$, vanishing outside $V_2(q)$. Then $v \doteq \overline{\partial}\, \tilde{s}$ is a global smooth section of $\wedge^{n,1} T^* X \otimes E$, with the following properties:

- $v\big|_{U(q) \cap V_1(q)} \equiv 0$
- $v\big|_{X \setminus V_2(q)} \equiv 0$

In particular since v is zero around q, it follows that, given a smooth function $\varphi \colon X \setminus \{q\} \to \mathbb{R}$, with $\varphi(z) = n \log|z - q|$ around q, we have

- $v \in L^2_\varphi(X, \wedge^{n,1} T^* X \otimes E)$

Therefore, by Theorem 3.24, we have that there exists $u \in L^2_\varphi(X, \mathcal{O}(K_X + E))$, such that $\overline{\partial}\, u = v$. Such a function u satisfies

$$\overline{\partial}\, u = 0 \textit{ near } q$$

and since $u \in L^2_\varphi$, this implies that $u_q = 0$.

To conclude take $\sigma \doteq \tilde{s} - u$. Clearly σ is a global section of $K_X + E$, and holomorphic since $\overline{\partial}\,\sigma = \overline{\partial}\,\tilde{s} - \overline{\partial}\, u = v - v = 0$; moreover $\sigma_q = \tilde{s}_q - u_q = s(q)$.

3.4.2 The Kodaira Embedding Theorem

Theorem 3.33 (Kodaira Embedding Theorem)**.** *If L is a positive line bundle over a compact complex manifold X, then L is ample.*

Proof. Let h_0 be a metric on L such that $i\Theta(L, h_0) > 0$. Define

$$E \doteq -K_X + mL$$

for some $m \in \mathbb{Z}^+$ to be chosen.

Since $-K_X$ and mL inherit metrics of X and L respectively, we have that E has natural induced metric, h, of curvature

$$\Theta(E, h) = Ric(\omega) + im\Theta(L, h_0)$$

Fix $p, q \in X$, and consider a correspondent log-like function $\varphi_{p,q}$.

Now, choose m such that

$$Ric(\omega) + im\Theta(L, h_0) + i\,\partial\,\overline{\partial}\,\varphi_{p,q} > 0$$
$$Ric(\omega) + im\Theta(L, h_0) > 0$$

Observe that such an m can be found, since $i\Theta(L, h_0) > 0$, X is compact, and therefore therefore $Ric(\omega)$ as well as $\partial\,\overline{\partial}\,\varphi_{p,q}$ are bounded. Then we can then apply

(3.31) and similarly (3.24) to get the surjective mappings:

$$H^0(X, \mathcal{O}(mL)) \overset{r_1}{\twoheadrightarrow} (mL)_p \oplus (mL)_q \tag{3.25}$$
$$H^0(X, \mathcal{O}(mL)) \overset{r_2}{\twoheadrightarrow} (mL)_p \otimes (\mathbb{C} \oplus T_p^* X) \tag{3.26}$$

For every p and q

Let $s_0, \cdots, s_d$ be a basis of $H^0(X, \mathcal{O}(mL))$ the holomorphic global sections of mL. In a local holomorphic trivialization of mL, $e\colon U(p) \to mL$, we have that $s_i(x) = g_i(x)e_x$ for some $g_i\colon U(p) \to \mathbb{C}$, and $x \in U(p)$. We define the following holomorphic map

$$\begin{aligned} \Phi\colon X &\to \mathbb{C}P^d \\ x &\mapsto [g_0(x) : \cdots : g_d(x)] \end{aligned}$$

This is well-defined since by (3.26) we can find $g_j(x) \neq 0$ for some j, and if $\tilde{e}\colon V(q) \to mL$ is a different trivialization, we have that there exists $f\colon V(q) \to \mathbb{C}^*$ holomorphic map such that $s_i(x) = g_i(x)f(x)\tilde{e}$ for every i, therefore

$$[g_0(x) : \cdots : g_d(x)] = [f(x)g_0(x) : \cdots : f(x)g_d(x)]$$

We will prove that Φ is an embedding.

For any given $p \in X$, by (3.26), we can find global holomorphic sections $\sigma_0, \cdots, \sigma_n \in H^0(X, \mathcal{O}(mL))$ such that in a local trivialization around p $\sigma_0(p) = 1$, $\sigma_i(p) = 0$, $d_p\sigma_j = dz_p^j$. It follows that the set $\{\sigma_i\}_{i=0}^n$ is linearly independent in $H^0(X, \mathcal{O}(mL))$, which implies that we can complete it to a base. Using this base in the definition of Φ, and computing it in the charts $(1, \zeta^1, \ldots, \zeta^n, \ldots)\colon V \subseteq \mathbb{C}P^d \to \mathbb{C}^{d+1}$ and $(z^1, \ldots, z^n)\colon U(p) \subseteq X \to \mathbb{C}^n$, we have that

$$\frac{\partial(\Phi_1, \ldots, \Phi_n)}{\partial(z^1, \ldots, z^n)} \neq 0$$

Which implies that Φ is an immersion.

It remains to prove that Φ is injective. Let $p, q \in X$, $p \neq q$, by (3.25) we have that we can find $\sigma_0, \sigma_1 \in H^0(X, \mathcal{O}(mL))$ such that $\sigma_0(p) = 0, \sigma_0(q) = 1$ and $\sigma_1(p) = 1, \sigma_1(q) = 0$, completing σ_0, σ_1 to a base of $H^0(X, \mathcal{O}(mL))$ and taking the corresponding Φ it becomes clear that Φ is injective.

For the proof to be complete we need to asses the following problem: Every time that we chose a basis of the global holomorphic sections, we changed the map Φ.

To solve this observe that if $\tilde{s}_0, \cdots, \tilde{s}_d$ and $s_0, \cdots, s_d$ are basis of $H^0(X, \mathcal{O}(mL))$, then there exists a matrix $A \in GL\big(H^0(X, \mathcal{O}(mL))\big) \cong GL(d+1, \mathbb{C})$ such that the vectors

$$\tilde{s} = As$$

Define the associated map

$$\begin{aligned} PA \colon \mathbb{C}P^d &\to \mathbb{C}P^d \\ [z^0 : \cdots : z^d] &\mapsto [A_{0,j} z^j, \cdots, A_{d,j} z^j] \end{aligned}$$

We then have that the following diagram is commutative

$$\begin{tikzcd} X \arrow[rr, "\Phi"] \arrow[dr, "\tilde{\Phi}"'] & & \mathbb{C}P^d \arrow[dl, "PA"] \\ & \mathbb{C}P^d & \end{tikzcd} \tag{3.27}$$

Where $\tilde{\Phi}$ is given by the basis $\tilde{s}_0, \cdots, \tilde{s}_d$. It is then clear that since PA is a biholomorphism, it follows that Φ is an embedding/immersion/injective if and only if $\tilde{\Phi}$ it also is.

The theorem holds, since an injective immersion defined on a compact is an embedding. $\square$

Appendix A

Functional Analysis

Definition A.1. *Let H be a Hilbert space. We will say that a linear map $T\colon D(T) \subset H \to H$ is an* operator *if the domain of T, $D(T)$, is dense in H.*

Given an operator T on the Hilbert space H, let us denote by $G(T)$ the *graph of* T, which is the linear subspace of the direct sum $H \oplus H$ defined by:

$$G(T) = \big\{(x, Tx) : x \in D(T)\big\}.$$

The operator T is said to be *closed* if $G(T)$ is a closed subspace of $H \oplus H$. By the closed graph theorem, it T is a closed operator whose domain is H, then T is bounded. An *extension* of the operator T is an operator S whose domain $D(S)$ cointains $D(T)$, and such that $Sx = Tx$ for all $x \in D(T)$. An operator is said to be *closable* if it admits a closed extension. It is not hard to show that T is closable if and only if the closure of $G(T)$ is a graph, i.e., iff there exists a (closed) operator S such that $G(S) = \overline{G}(T)$. In this situation, S is called the *closure of* T, denoted by $\overline{T}$, and this is the *smallest* closed extension of T.

Definition A.2. *Let T be an operator on H, and let*

$$D(T^*) \doteq \{v \in H : \exists\, w_v \in H \textit{ such that } \langle Tu, v\rangle = \langle u, w_v\rangle\ \forall u \in D(T)\} \tag{A.1}$$

Define $T^*(v) \doteq w_v$, the density of $D(T)$ guarantees that there are no choices involving *the definition of T^*, that is that w_v is unique vector satisfying A.1.*

Using the very definition of the adjoint T^*, it is easy to show that the following equality holds:

$$G(T^*) = \big[J\big(G(T)\big)\big]^\perp, \tag{A.2}$$

where $J\colon H \oplus H \to H \oplus H$ is defined by $J(x,y) = (-y,x)$ for all $x,y \in H$. From (A.2) we obtain easily that $G(T^*)$ is always a closed. Note that J is a linear isometry; in particular:

$$G(T^*)^\perp = J\big(G(T)\big), \qquad \text{if } T \text{ is closed.} \tag{A.3}$$

Observe also that J is a complex structure on $H \oplus H$, i.e., $J^2 = -\mathrm{Id}$.

Theorem A.3. *If T is a closed operator, then T^* is densely defined, and therefore it is a closed operator. In this case, $(T^*)^* = T$. Moreover, the following hold:*

(a) $\ker T^* = (\mathrm{Im} T)^\perp$

(b) $(\ker T)^\perp = \overline{\mathrm{Im} T^*}$

Proof. In order to show that $D(T^*)$ is dense, we need to show that if $x_0 \in \big[D(T^*)\big]^\perp$, then $x_0 = 0$. Since $x_0 \in \big[D(T^*)\big]^\perp$, $(x_0, 0) \in \big[G(T^*)\big]^\perp = J\big(G(T)\big)$. Thus:

$$(0, x_0) = J(x_0, 0) \in J^2\big(G(T)\big) = -G(T) = G(T),$$

which implies $x_0 = 0$. Now, it is easy to see that $(T^*)^*$ is an extension of T. Using (A.2) repeatedly, and the fact that J is an isometry with $J^2 = -\mathrm{Id}$, we obtain:

$$G\big((T^*)^*\big) = \big[J\big(G(T^*)\big)\big]^\perp = J\big([G(T^*)]^\perp\big) \overset{\text{by (A.3)}}{=} J^2\big(G(T)\big) = G(T),$$

which implies $(T^*)^* = T$.

By definition:

$$\ker(T^*) = \big\{y \in H : \langle Tx, y\rangle = 0,\ \forall\, x \in D(T)\big\} = (\mathrm{Im}\, T)^\perp,$$

which proves (a).

Now, using the fact that $(T^*)^* = T$, from (a) we obtain:

$$(\ker T)^{\perp} = \ker\left((T^*)^*\right)^{\perp} = \left(\operatorname{Im} T^*)^{\perp}\right)^{\perp} = \overline{\operatorname{Im} T^*},$$

which proves (b). $\square$

Bibliography

[1] José Bertin, Jean-Pierre Demailly, Luc Illusie, and Chris Peters. *Introduction à la théorie de Hodge*, volume 3 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 1996.

[2] Sebastienne Boucksom. *Singularities of Plurisubharmonic Functions and Multiplier Ideals*. Published online, 2017. `http://sebastien.boucksom.perso.math.cnrs.fr/notes/L2.pdf`.

[3] N. Bourbaki, H.G. Eggleston, and S. Madan. *Topological Vector Spaces: Chapters 1–5*. Elements of Mathematics. Springer Berlin Heidelberg, 2002.

[4] John B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[5] Jean-Pierre Deamilly. *Complex analytic and differential geometry*. Published online, 2007. `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.102.2767&rep=rep1&type=pdf`.

[6] T Gantumur. Introduction to distributions. Lecture notes available on webpage at `https://www.math.mcgill.ca/gantumur/math581w14/distrib.pdf`.

[7] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[8] Daniel Huybrechts. *Complex geometry*. Universitext. Springer-Verlag, Berlin, 2005. An introduction.

[9] Shoshichi Kobayashi. *Differential geometry of complex vector bundles*. Princeton Legacy Library. Princeton University Press, Princeton, NJ, [2014]. Reprint of the 1987 edition [MR0909698].

[10] Steven Rosenberg. *The Laplacian on a Riemannian manifold*, volume 31 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. An introduction to analysis on manifolds.

[11] F. Treves. *Topological Vector Spaces, Distributions and Kernels*. Dover books on mathematics. Dover Publications, 2006.

[12] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2008. With a new appendix by Oscar Garcia-Prada.