# A NON-ARCHIMEDEAN THEORY OF COMPLEX SPACES AND THE CSCK PROBLEM

### PIETRO MESQUITA-PICCIONE

ABSTRACT. In this paper we develop an analogue of the Berkovich analytification for non-necessarily algebraic complex spaces. We apply this theory to generalize to arbitrary compact Kähler manifolds a result of Chi Li, [Li22], proving that a stronger version of K-stability implies the existence of a unique constant scalar curvature Kähler metric.

### CONTENTS

1
6
11
28
33
44
51
54
57
58
58
60

### INTRODUCTION

The conjecture. The study of algebraic and analytic properties of complex manifolds has been an active field of research in the area of Kähler Geometry.

To approach many of the questions in this field, one requires generally results and techniques from both Differential and Algebraic Geometry, which shows the connection between those fields.

One of the most important questions in the area is the Yau–Tian–Donalson (YTD) conjecture. It deals with the following question:

**Question.** Let  $(X, \omega)$  be a Kähler manifold, and  $\alpha \doteq [\omega]$  be the cohomology class of  $\omega$ . Is there a Kähler metric  $\omega' \in \alpha$ , in the same class as  $\omega$ , such that its scalar curvature

$$\operatorname{Scal}(\omega') \equiv \underline{s}$$
 (0.0.1)

is constant?

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 94532.

When X is a Fano variety, i.e. X is anticanonically polarized, Chen–Donaldson–Sun [CDS15a, CDS15b, CDS15c] and Tian [Tia15] showed, using the continuity method, that X admits a constant scalar curvature Kähler metric (cscK metric) if, and only if, X is K-stable. Later other proofs became available, like [BBJ21], [Szé16, DS16], [CSW18], and [DZ24] among others.

K-stability is an algebro-geometric notion that deals with the study of 1-parameter degenerations of the manifold X, together with a numerical invariant. This notion is an infinite dimensional analogue of stability in the sense of Geometric Invariant Theory (GIT). In that context, by the the Kempf–Ness Theorem, studying the stability of an orbit of a hamiltonian group action is equivalent to finding zeros of an associated moment map.

The general –stil open– version of the Yau–Tian–Donaldson conjecture reads:

**Conjecture** (YTD Conjecture). There exists a unique constant scalar curvature Kähler (cscK) metric in  $\alpha$  if, and only if,  $(X, \alpha)$  is uniformly K-stable in the sense of [SD18, DR17b].

A variational approach. The PDE of equation (0.0.1) has a variational interpretation that will play the key role in our approach to the YTD Conjecture. This approach was developed by many authors, see [BB17], [BBE+19], [Che00], [Dar15], and [DR17a] for some important work. We describe this approach now.

By the  $\partial \overline{\partial}$ -lemma, if  $\omega' \in \alpha$  is a Kähler metric in the cohomology class of  $\omega$ , then  $\omega' - \omega = \mathrm{dd}^{c} u$ , for  $u: X \to \mathbb{R}$  a smooth function satisfying:

$$\omega + \mathrm{dd}^{\mathrm{c}} u > 0.$$

We denote  $\omega'$  by  $\omega_u$ , and we call u the *potential of*  $\omega'$ . In these terms, Equation (0.0.1) translates to a fourth order elliptic PDE on the potential.

The space of the potentials, in a given class  $[\omega]$ , is denoted by

$$\mathcal{H}(\omega) \doteq \{ u \colon X \to \mathbb{R} \mid u \text{ smooth, and } \omega + \mathrm{dd}^{\mathrm{c}} u > 0 \}.$$

In order to have a good geometric setting to carry out the variational calculus for studying this PDE, we must consider a completion<sup>1</sup> of  $\mathcal{H}(\omega)$ , the space  $\mathcal{E}^{1}(\omega)$ .

By the groundbreaking work of Chen–Cheng [CC21a, CC21b], we know that there exists a unique solution for (0.0.1) if and only if a given functional, called the *Mabuchi* functional,  $M_{\omega}: \mathcal{E}^{1}(\omega) \to \mathbb{R}$ , satisfies:

$$\mathbf{M}_{\omega} \ge \delta \mathbf{J}_{\omega} - C, \tag{0.0.2}$$

for some  $C, \delta > 0$ , and where  $J_{\omega}$  is a "norm like" functional. If it is the case,  $M_{\omega}$  is said to be *coercive*.

In this setting the YTD conjecture can be stated as follows:

**Conjecture** (YTD Conjectrue). The Mabuchi functional is coercive if, and only if,  $(X, \alpha)$  is uniformly K-stable in the sense of [SD18, DR17b].

**Remark 0.0.1.** When X is a Kähler variety of klt singularities and admits a resolution of singularities of nef relative anticanonical bundle, by combining works of [BJT24, PT24], we have that the coercivity of the Mabuchi functional implies the existence of a positive closed current in  $\alpha$ , that is a cscK metric on the regular locus, and has bounded potential. In this generality, the above conjecture is completly open.

<sup>&</sup>lt;sup>1</sup>for the Darvas metric  $d_1$ , see [Dar15].

3

Varional approach in the Fano setting. In the Fano case, Berman–Boucksom–Jonsson, in [BBJ21], used the variational approach to prove the conjecture. Their proof relied on two observations:

- It was already known, in the Fano case, that the existence of a cscK metric is equivalent to the *coercivity* of a different –and simpler– functional defined on the space of potentials  $D: \mathcal{E}^1(\omega) \to \mathbb{R}$ , the *Ding functional*.
- By [BHJ17], there is a non-archimedean description of K-stability: there is a nonarchimedean counterpart of  $\mathcal{H}(\omega)$  on the *Berkovich analytification* of X, X<sup>an</sup>, denoted  $\mathcal{H}^{NA}$ , in which we can define a non-archimedean analogue of the Ding functional,  $D^{NA}: \mathcal{H}^{NA} \to \mathbb{R}$ , and K-stability becomes equivalent to the coercivity of this functional.

Thus, using the simpler analysis of the Ding functional, Berman–Boucksom–Jonsson prove that the slope formulas for psh rays of *algebraic singularities* guarantee the coercivity of the functional D, as soon as one supposes the coercivity of the non-archimedean counterpart  $D^{NA}$ .

In the general case the conjecture is still open, [SD18] and [DR17b] independently proved -in the Kähler case- one direction of the conjecture: the coercivity of the Mabuchi functional implies uniform K-stability.

The best result in the reciprocal direction is due Chi Li: in [Li22], he adapted the strategy of Berman–Boucksom–Jonsson to the Mabuchi functional, getting a weaker form of the open direction of the YTD conjecture for projective manifolds, i.e. he proves that stronger version of K-stability implies the coercivity of the Mabuchi functional.

The key ingredient in Chi Li's paper consists of proving that a distabilizing ray, a ray of functions that contradicts the coercivity of the Mabuchi functional  $M_{\omega}$ , must be a maximal geodesic ray, that is, it must come from a non-archimedean potential of finite energy: Like for  $\mathcal{H}(\omega)$ , the set of potentials of finite energy  $\mathcal{E}^1(\omega)$  also has a non-archimedean counterpart,  $\mathcal{E}^{1,NA}$ , and there is a correspondence between such potentials of finite energy and maximal geodesic rays, a disguished class of rays in  $\mathcal{E}^1(\omega)$ . Having this in hand, he uses the –finer– slope formulas for maximal rays, and then the strategy of [BBJ21] follows in this general polarized case.

The goal of this paper is to generalize this result of Chi Li to the transcendental setting. We prove:

**Theorem A.** Let  $(X, \alpha)$  be a compact Kähler manifold that is uniformly K-stable over  $\mathcal{E}^1$ . Then,  $\alpha$  contains a unique cscK metric.

**General strategy and main results.** By the work of Sjöström Dyrefelt and Dervan–Ross, there exists a theory of K-stability for Kähler manifolds, but in order to adapt the strategy of Chi Li –or ultimately of BBJ–, one needs a non-archimedean theory for a transcendental complex manifold, where the language of K-stability can be translated to.

We hence start by developping this non-archimedean theory. More precisely, we associate to a complex manifold, X, a "non-archimedean" compact Hausdorff topological space of *semivaluations* on X, defined as the *tropical spectrum* of the set of coherent ideals of X, which we denote by  $X^{\beth}$ . This notion coincides with the Berkovich analytification of X whenever the latter is a proper algebraic variety<sup>2</sup>, and moreover it also has a description as the limit of a Dual Complex, just like in algebraic setting.

<sup>&</sup>lt;sup>2</sup>We recall that if X is a proper scheme over a trivially valued field, its Berkovich analytification  $X^{\operatorname{an}}$  is defined as the set of all valuations  $v \colon \mathbb{C}(Y)^* \to \mathbb{R}$ , for  $Y \subseteq X$  a subvariety, and  $\mathbb{C}(Y)$  its function field. When X is a compact complex manifold, we may have that its function field is trivial, and moreover it may not have any non-trivial subvarieties. We cannot use this approach to define  $X^{\square}$ .

The most important class of semivaluations is the one of the *divisorial valuations*, which are given by

$$\mathcal{O}_X \supseteq I \mapsto \operatorname{ord}_F(I \cdot \mathcal{O}_Y)$$

for  $F \subseteq Y \to X$  a smooth irreducible reduced divisor on a normal model of X. The next main theorem we prove establishes the density of the set of divisorial valuations,  $X^{\text{div}}$ , on  $X^{\beth}$ .

# **Theorem B.** $X^{\text{div}}$ is dense in $X^{\beth}$ .

The proof of Theorem B crucially requires a description of the divisorial valuations of X as the  $\mathbb{C}^*$ -invariant divisorial valuations on  $X \times \mathbb{P}^1$ . We get this description by studying the divisorial valuations on "local algebraic models" of X, that is, for each  $p \in X$ , we study the divisorial valuations of the scheme  $X_p \doteq \operatorname{Spec} \mathcal{O}_{X,p}$ .

After establishing some first basic results, we develop a non-archimedean pluripotential theory for  $X^{\beth}$ , key for the non-archimedean approach of [BBJ21]. We define nonarchimedean psh functions, and a mixed energy coupling, that allow us to:

- (1) Use the synthetic pluripotential theory of [BJ23].
- (2) Define non-archimedean versions of classical functionals like: the *Monge–Ampère* energy; the twisted energy; the entropy; and finally the J functional.

We then denote by  $\mathcal{E}^1(\alpha)$  the set of non-archimedean potentials of finite Monge-Ampère energy.

In the algebraic setting this theory coincides with the one of [BJ22]. In the transcendental setting Darvas–Xia–Zhang develop a non-archimedean pluripotential theory for a big class, on [DXZ23], that coincides with ours on a Kähler class. Their non-archimedean pluripotential theory is not over a "non-archimedean space", and hence ours is a more direct analogue of the algebraic theory. In particular, we can make sense of Monge–Ampère equations in our case, while it is not clear how to interpret it in their formalism.

Like in [BBJ21], we have that the pluripotential theory developed here behaves well with the complex one: to each psh ray of potentials  $U_t$  we have associated a non-archimedean psh function  $U^{\neg}$ . Reciprocally, to each non-archimedean psh function of finite energy  $\varphi \in \mathcal{E}^1(\alpha)$ , there exists a psh ray  $V_t$  such that  $V^{\neg} = \varphi$ . Moreover, there exists a 1to1 correspondence between maximal geodesic rays in  $\mathcal{E}^1(\omega)$  and non-archimedean potentials in  $\mathcal{E}^1(\alpha)$ . This is the basis of the correspondence with the theory of [DXZ23].

For maximal geodesic rays we get formulas of the type:

$$\lim_{s \to \infty} \frac{\mathcal{F}(U_s)}{s} = \mathcal{F}^{\beth}(U^{\beth}), \qquad (0.0.3)$$

for F either E,  $E^{\eta}$ , or J, and an inequality for the entropy:

$$\lim \frac{\mathrm{H}_{\omega}(U_s)}{s} \ge \mathrm{H}_{\alpha}(U^{\beth}). \tag{0.0.4}$$

Adding all together we get:

$$\lim_{s \to \infty} \frac{\mathcal{M}_{\omega}(U_s)}{s} \ge \mathcal{M}_{\alpha}(U^{\beth}),$$

where  $M_{\alpha}$  is the non-archimedean Mabuchi functional, the non-archimedean counterpart of  $M_{\omega}$  providing the good inequality to conclude the proof of Theorem A.

Indeed, just like in the projective setting, if  $M_{\omega}$  is not coercive we can find a geodesic ray  $U_t \in \mathcal{E}^1$  such that  $t \mapsto M_{\omega}(U_t)$  is decreasing for t large. We call such a ray a *destabilizing* geodesic ray.

Analogously to [Li22], every destabilizing ray is maximal, that is, it is a geodesic ray coming from a non-archimedean potential of finite energy,  $\varphi \in \mathcal{E}^1(\alpha)$ . In particular:

$$\lim_{t \to \infty} \frac{\mathcal{M}_{\omega}(U_t)}{t} \ge \mathcal{M}_{\alpha}(\varphi). \tag{0.0.5}$$

Like this, we prove that if  $(X, \omega)$  is such that  $M_{\alpha} \colon \mathcal{E}^{1}(\alpha) \to \mathbb{R}$  is positive, then there exists a unique cscK metric on  $\alpha$ , proving Theorem A.

The YTD conjecture and Chi Li's result. In the non-archimedean terms of the present paper K-stability reads as the coerciviness of the non-archimedean Mabuchi functional:

**Definition 0.0.2.** The pair  $(X, \alpha)$  is uniformly K-stable if there exists a  $\delta > 0$  such that

$$M_{\alpha}(\varphi) \ge \delta J_{\alpha}(\varphi)$$

for every  $\varphi \in \mathcal{H}(\alpha)$ , that is if  $M_{\alpha}$  is coercive over  $\mathcal{H}(\alpha)$ .

We have proved that a stronger version of K-stability, uniform K-stability over  $\mathcal{E}^{1}(\alpha)$ , implies the coerciveness of the K-energy, and hence the existence of an unique cscK metric. In order to prove the conjecture one would need to prove that

$$M_{\alpha}|_{\mathcal{H}(\alpha)} \ge \delta J_{\alpha}|_{\mathcal{H}(\alpha)} \implies M_{\alpha}|_{\mathcal{E}^{1}(\alpha)} \ge \delta' J_{\alpha}|_{\mathcal{E}^{1}(\alpha)}$$

for some  $\delta, \delta' > 0$ .

In [Li22], Chi Li proves that, in the projective case, one has

$$M_{\alpha}|_{CPSH(\alpha)} \ge \delta J_{\alpha}|_{CPSH(\alpha)} \implies M_{\alpha}|_{\mathcal{E}^{1}(\alpha)} \ge \delta J_{\alpha}|_{\mathcal{E}^{1}(\alpha)}$$

and hence he has an intermediate version of K-stability that implies the coerciveness of the K-energy.

We were not able to get a sharper result, like this one of Chi Li, in the transcendental setting. His result relies on the solution of the non-archimedean Monge–Ampère equation of [BFJ15] that we don't have at our disposal.

Furthermore, in the same paper Chi Li also gives a version of his theorem when X admits automorphisms, which we don't treat in present paper.

### Organization of the paper.

- In Section 1 we define our "non-archimedean analytification" of a locally ringed space, we do some basic constructions and we compare it to Berkovich analytification of a scheme.
- Section 2 deals with the study of  $X^{\neg}$ , when X is a normal compact complex space. We prove Theorem B, getting a theory which resembles closely the one of the Berkovich analytification of a projective variety over  $\mathbb{C}$ .
- Section 3 contains a dual complex description of  $X^{\beth}$ , analogue to the one found in [BJ22].
- Section 4 is devoted to developing the non-archimedean pluripotential theory of X<sup>□</sup>.
- In Section 5 we compare the pluripotential theory of  $X^{\neg}$  with the transcendental non-archimedean theory of [DXZ23, Xia24], and get slope formulas, generalizing the results of [SD18, DR17b, Li22].
- In Section 6 we define *strong K-stability*, and prove Theorem A, the main result of the paper.

Acknowledgements. I would specially like to thank my PhD advisors Sébastien Boucksom and Tat Dat Tô for all their generous help, support and patience while discussing the present article for many months.

I would also like to thank Mingchen Xia for great conversations and advice, and all my friends for their support.

### NOTATIONS AND CONVENTIONS

Through out this paper, a *ring* will always be unital and commutative. By an *analytic* variety we mean a reduced and irreducible complex analytic space. Given X an analytic variety we denote by  $\widetilde{X}$  the normalization of X, cf. [GR12, Chapter 6]. Moreover, we simply call *ideal* a coherent ideal sheaf of  $\mathcal{O}_X$ . We call *flag ideal* a  $\mathbb{C}^*$ -invariant fractional coherent ideal of  $\mathcal{O}_{X \times \mathbb{P}^1}$ , supported on  $X \times \{0\}$ , and we denote the set of such ideals by  $\mathcal{F}$ .

A flag ideal  $\mathfrak{a} \in \mathcal{F}$  can be written as a sum:

$$\mathfrak{a} = \sum_{\lambda \in \mathbb{Z}} \mathfrak{a}_{\lambda} t^{\lambda}, \tag{0.0.6}$$

for  $(\mathfrak{a}_{\lambda})_{\lambda}$  an increasing sequence of ideals of X such that  $\mathfrak{a}_{\lambda} = 0$  for  $\lambda \ll 0$  and  $\mathfrak{a}_{\lambda} = \mathcal{O}_X$  for  $\lambda \gg 0$ .

An ideal on X will typically be denoted by I, J or K. An ideal on  $X \times \mathbb{P}^1$  will typically be denoted either  $\mathfrak{a}$  or  $\mathfrak{b}$ .

An ideal on X I is a *prime ideal* if given J, K ideals on X satisfying:

 $I \supseteq J \cdot K,$ 

then either  $I \supseteq J$ , or  $I \supseteq K$ . An ideal is prime if, and only if, it is a radical ideal and the underlying analytic subspace is irreducible. That is, prime ideals of X are exactly the ideals attached to analytic subvarieties.

Let X be an analytic variety, I a coherent ideal, and  $q \in \mathbb{Q}$ , a map  $g: X \to [-\infty, +\infty[$  has singularities of type  $I^q$  if locally:

$$g(z) = q \log \sum_{i=1}^{k} |f_i| + O(1),$$

for  $f_1, \ldots, f_k$  local generators of I.

If X is a Kähler manifold, we denote by  $\mathcal{K}(X)$  the set of Kähler forms on X, by  $\operatorname{Pos}(X) \subseteq H^{1,1}(X)$  the set of Kähler classes, and  $\operatorname{Nef}(X)$  the set of nef classes, i.e. the closure  $\overline{\operatorname{Pos}(X)} \subseteq H^{1,1}(X)$ . Moreover, if we have fixed a holomorphic map  $f: X \to Y$ , we denote by  $\operatorname{Pos}(X/_f Y) = \operatorname{Pos}(X/Y)$  the set of classes which are f-relatively Kähler, and analogously  $\operatorname{Nef}(X/_f Y) = \operatorname{Nef}(X/Y)$  the set of nef classes relatively to Y.

For  $\omega \in \mathcal{K}(X)$ , we denote by  $\operatorname{Ric}(\omega)$  its *Ricci form*. The trace of  $\operatorname{Ric}(\omega)$  is denoted by  $\operatorname{Scal}(\omega)$ , the *scalar curvature*, and the cohomological quantity:

$$n \cdot \frac{[\operatorname{Ric}(\omega)] \cdot [\omega]^{n-1}}{[\omega]^n},$$

the average of the scalar curvature, by  $\underline{s}$ .

## 1. Berkovich spectra as tropical spectra

1.1. Berkovich spectrum. Let X be an affine scheme, that is  $X = \operatorname{Spec} R$  for some ring R.

If we consider R as a normed ring, with the trivial norm (the norm that to  $a \in R$  associates  $||a||_{\text{triv}}$  that is 1 if  $a \neq 0$ , and 0 otherwise), then R can be seen as a Banach

ring. In particular, we can associate to it a compact Hausdorff topological space, the *Berkovich spectrum of* R, first defined in [Ber90].

**Definition 1.1.1.** The Berkovich spectrum associated to  $(R, \|\cdot\|_{triv})$ , denoted by  $\mathcal{M}(R)$ , is the set of bounded multiplicative semi-norms on R, i.e.

$$\mathbf{N} \in \mathcal{M}(R) \iff N \le \|\cdot\|_{\mathrm{triv}},$$

equipped with the Hausdorff topology of pointwise convergence.

The topology of pointwise convergence is the subspace topology given by natural inclusion

$$\mathcal{M}(R) \subseteq \prod_{a \in R} [0, 1], \tag{1.1.1}$$

as an easy consequence of Tychonoff's theorem  $\mathcal{M}(R)$  is compact.

To a semi-norm  $N \in \mathcal{M}(R)$  we can attach a *semivaluation* on R, by taking  $-\log N$ . Thus, equivalently, we can consider the Berkovich spectrum of R as the set of semivaluations of R with values on  $[0, +\infty]$ , which, following the notation of [Thu07], we will denote by  $X^{\exists}$ , that is:

$$X^{\beth} \doteq \left\{ \begin{array}{c} v \colon R \to [0, +\infty] \\ v \colon R \to [0, +\infty] \end{array} \middle| \begin{array}{c} v(a \cdot b) = v(a) + v(b) \\ v(a + b) \ge \min\{v(a), v(b)\} \\ v(1) = 0 & v(0) = +\infty \end{array} \right\}.$$
 (1.1.2)

The construction is functorial, that is, given Y another affine scheme and a morphism  $f: Y \to X$ , we can associate a continuous map between  $Y^{\beth}$  and  $X^{\beth}$ :

$$f^{\beth} \colon Y^{\beth} \to X^{\beth}$$
$$v \mapsto f^{\beth}(v) \colon a \mapsto v(f(a)),$$

compatible with compositions.

The Berkovich spectrum comes with a natural class of continuous functions. Given  $a \in R$ , we associate  $|a|: \mathcal{M}(R) \to \mathbb{R}_+$  by the formula  $N \mapsto N(a)$ , or equivalently:

$$\log|a| \colon X^{\perp} \to [0, +\infty]$$
$$v \mapsto -v(a).$$

The notation is so that  $\exp(\log|a|) = |a|$ .

There is an equivalent formulation of Berkovich spectrum of a trivially normed ring, namely the *tropical spectrum* of the semi-ring of its ideals of finite type.

Now we study such an object.

1.2. Tropical spectrum. For more details on this section see Appendix A.

**Definition 1.2.1.** Let  $(S, +, \cdot)$  be a semi-ring, and consider the semi-ring of the extended real line  $(]-\infty, +\infty]$ , min, +), we define the tropical spectrum of S as the topological space given by the set of tropical characters, *i.e.* the semi-ring morphisms from S to  $]-\infty, +\infty]$ :

TropSpec 
$$S \doteq \left\{ \begin{array}{c} \chi \colon S \to \left] -\infty, +\infty \right] \\ \chi(a \cdot b) = \chi(a) + \chi(b) \\ \chi(0) = +\infty \\ \chi(a + b) = \min\{\chi(a), \chi(b)\} \end{array} \right\},$$
 (1.2.1)

endowed with the pointwise convergence topology. Moreover, if S has a multiplicative identity we ask  $\chi(1) = 0$ , this is equivalent  $\chi$  not identically  $+\infty$ .

As before, with this topology  $\operatorname{TropSpec} S$  is compact and Hausdorff.

The tropical spectrum comes with a natural order relation, and a natural order compatible  $\mathbb{R}_{>0}$ -action, the usual order of functions, and multiplication by scalar action. Moreover, the  $\mathbb{R}_{>0}$  action on  $X^{\beth}$ , induces an action on  $\mathbb{C}^{0}(X^{\beth}, \mathbb{R})$ , for  $t \in \mathbb{R}_{>0}$  we define:

$$(t \cdot \varphi)(v) = t\varphi(t^{-1}v). \tag{1.2.2}$$

**Remark 1.2.2.** For any semi-ring S, TropSpec S is non-empty. Indeed, we define  $\chi_{\text{triv}}: S \to [0, +\infty]$ , by the formula:

$$\chi_{\text{triv}}(a) = 0, \quad \text{for every } a \in S \setminus \{0\}.$$

It is easy to see that  $\chi_{triv}$  is a semivaluation, and, moreover, it is a fixed point of the  $\mathbb{R}_{>0}$ -action.

1.2.1. Comparison with the Berkovich spectrum. Now, let R be a ring, and  $\mathscr{I}(R)$  be the set of ideals of finite type of R. Together with the usual operations of sum and product  $\mathscr{I}(R)$  can be seen as a semi-ring. Moreover, as an easy consequence of the algebraic structure of  $\mathscr{I}(R)$ , its tropical characters are all positive, that is:

TropSpec 
$$\mathscr{I}(R) = \left\{ \begin{array}{c} \chi \colon \mathscr{I}(R) \to [0, +\infty] \\ \chi \colon \mathscr{I}(R) \to [0, +\infty] \end{array} \middle| \begin{array}{c} \chi(I \cdot J) = \chi(I) + \chi(J) \\ \chi(I + J) = \min\{\chi(I), \chi(J)\} \\ \chi(R) = 0 \& \chi(0) = +\infty \end{array} \right\}$$

see Appendix A for more details.

There is a natural continuous map:

TropSpec 
$$\mathscr{I}(R) \to (\operatorname{Spec} R)^{\perp}$$
,

that assigns to each tropical character  $\chi \in \operatorname{TropSpec} \mathscr{I}(R)$  the semivaluation:

$$R \ni a \mapsto \chi(a \cdot R).$$

**Proposition 1.2.3.** The natural map,  $\operatorname{TropSpec} \mathscr{I}(R) \to (\operatorname{Spec} R)^{\beth}$ , is a homeomorphism.

*Proof.* Since TropSpec  $\mathscr{I}(R)$  is compact it is enough to check that the map is bijective.

The desired inverse function,  $(\operatorname{Spec} R)^{\square} \to \operatorname{TropSpec} \mathscr{I}(R)$ , is the one that assigns to  $v \in (\operatorname{Spec} R)^{\square}$  the character:

$$\mathscr{I}(R) \ni I \mapsto \min_{f \in I} v(f),$$

where the minimum is achieved on any (finite) set of generators.

With this "tropical" characterization of the Berkovich spectrum of a ring, we will extend this construction to locally ringed spaces.

1.3. Semivaluations on locally ringed spaces. Let  $(X, \mathcal{O}_X)$  be a locally ringed topological space, and denote by  $\mathscr{I}_X$  the set of  $\mathcal{O}_X$ -ideals locally of finite type. The set  $\mathscr{I}_X$  has a semi-ring structure given by the usual addition and multiplication of ideal sheaves, and we define:

**Definition 1.3.1.** Let X be a locally ringed space, we define the space of semivaluations on X as the tropical spectrum of  $\mathscr{I}_X^3$ :

$$X^{\beth} \doteq \operatorname{TropSpec} \mathscr{I}_X = \left\{ \begin{array}{c} v \colon \mathscr{I}_X \to [0, +\infty] \\ v \colon \mathscr{I}_X \to [0, +\infty] \end{array} \middle| \begin{array}{c} v(I \cdot J) = v(I) + v(J) \\ v(I + J) = \min\{v(I), v(J)\} \\ v(\mathcal{O}_X) = 0 \& v(0_X) = +\infty \end{array} \right\}$$
(1.3.1)

 $^{3}$ Again by Appendix A we have that all the tropical characters are positive.

Some geometrically relevant examples are as follows:

- **Example 1.3.2.** (1) If X is a scheme of locally finite type over a field k, equiped with the trivial absolute value,  $X^{\Box}$  is a subset of its Berkovich analytification,  $X^{an}$ , the set of semivaluations centered on X. This construction goes back to [Ber90] and [Thu07]. Whenever the scheme is proper, by the valuative criterion of properness  $X^{\Box} = X^{an}$ .
  - (2) If X is a complex analytic space, X<sup>□</sup> is an analogue of the Berkovich analytification of algebraic varieties over C. The study of X<sup>□</sup> will be done on Section 2, and will be the central object of study of the present paper.
  - (3) If X is a Berkovich space we can also associate to it an "analytified"  $X^{\beth}$ .

Here again the construction is functorial: a morphism  $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ , induces a mapping:

$$f^* \colon \mathscr{I}_Y \to \mathscr{I}_X$$

which in turn induces a continuous map:

$$f^{\beth} \colon X^{\beth} \to Y^{\beth}.$$

**Remark 1.3.3.** If X is a proper algebraic variety over  $\mathbb{C}$ , the GAGA theorems –for the Berkovich and complex analytifications– allow us to compare  $(X^{\mathrm{an}})^{\neg}$ ,  $(X_{\mathrm{hol}})^{\neg}$  and  $X^{\neg}$ , where  $X_{\mathrm{hol}}$  denotes the usual complex analytification.

Indeed, the theorems provide us with morphisms of ringed spaces between  $X_{hol}$ , X and  $X^{an}$ , which induce a 1to1 correspondence on the set of coherent ideal sheaves. Therefore we have canonical homeomorphisms:

$$(X^{\mathrm{an}})^{\beth} \simeq X^{\beth} \simeq (X_{\mathrm{hol}})^{\beth}$$

Moreover, as explained before, for a proper  $\mathbb{C}$ -scheme  $X^{an}$  coincides with  $X^{\beth}$ .

1.3.1. Properties of the functor  $\beth$ . Let X be a locally ringed space, and  $I \subseteq \mathcal{O}_X$  be an ideal of locally finite type. Consider  $Y \doteq \operatorname{supp}(\mathcal{O}_X/I) \subseteq X$ , together with the sheaf  $\mathcal{O}_Y \doteq (\mathcal{O}_X/I)|_Y$  We thus have that the inclusion

$$i: (Y, \mathcal{O}_Y) \hookrightarrow (X, \mathcal{O}_X)$$

is a ringed space morphism, and moreover we have the following lemma.

**Proposition 1.3.4.** Let  $Y \subseteq X$  as above, then

 $i^{\beth} \colon Y^{\beth} \to X^{\beth}$ 

is an embedding.

*Proof.* Since  $Y^{\beth}$  is compact it is enough to prove that  $i^{\beth}$  is injective.

By definition, the morphism

$$i^* \colon \mathcal{O}_X \twoheadrightarrow \mathcal{O}_Y$$

is surjective.

Therefore, if  $v, u \in Y^{\exists}$ , with  $v \neq u$ , then there exists an ideal locally of finite type  $J \in \mathscr{I}_Y$  such that  $v(J) \neq u(J)$ . Moreover,  $K \doteq (i^*)^{-1}(J) \subseteq \mathcal{O}_X$  is an ideal of locally finite type, and

$$i^{\beth}v(K) = v(J) \neq u(J) = i^{\beth}u(K).$$

**Remark 1.3.5.** We have just seen that the  $\beth$  functor preserves embeddings, however it does not preserve open mappings.

In point of fact, if  $U \subseteq X$  then  $U^{\exists}$  will be a compact subset of  $X^{\exists}$ , even if U is open on X.

1.3.2. Examples of semivaluations in  $X^{\beth}$ .

**Example 1.3.6.** Let  $p \in X$ , we denote by  $X_p$  the affine scheme  $\text{Spec } \mathcal{O}_{X,p}$ .

We have a natural continuous map from the set of local semivaluations at p to the set of global semivaluations on X, that to each  $v \in X_p^{\neg}$  assigns the valuation:

$$\mathscr{I}_X \ni I \mapsto v_p(I) \doteq v(I_p)$$

where  $I_p$  denotes the stalk of I at p.

**Remark 1.3.7.** If we suppose that the local ring  $\mathcal{O}_{X,p}$  is noetherian (or at least its maximal ideal of finite type), the map

$$v \mapsto v_p$$

is injective on the set of semivaluations centered at p, that is the set of semivaluations such that  $v(m_p) > 0$ .

Indeed, if v, v', centered at p, are such that there exists an ideal  $J \subseteq \mathcal{O}_{X,p}$  with

$$v(J) \neq v'(J),$$

then, for every  $k \in \mathbb{N}$  consider the ideal

$$J_k \doteq J + m_p^k \subseteq \mathcal{O}_{X,p}.$$

Since  $J_k$  is primary at p it extends to a global ideal, by triviality at any other point. Hence

$$v_p(J_k) = v(J_k) = \min\{v(J), k \cdot v(m_p)\}$$
  
$$v'_p(J_k) = v'(J_k) = \min\{v'(J), k \cdot v'(m_p)\},$$

for every k. Letting  $k \to +\infty$ , it follows that

$$\lim_{k \to \infty} v_p(J_k) = v(J), \quad \lim_{k \to \infty} v'_p(J_k) = v'(J).$$

We thus have  $v_p(J_k) \neq v'_p(J_k)$  for  $k \gg 1$ , and hence  $v_p \neq v'_p$ .

1.4. **PL functions.** Just like in the affine case, the space  $X^{\beth}$  comes with a natural class of functions, which we will call the *piecewise linear functions*. The terminology will become clear in Section 3.1.

**Definition 1.4.1.** Let  $I \in \mathscr{I}$  be an ideal locally of finite type, we have a function  $\log |I|: X^{\beth} \to [-\infty, 0]$  that maps:

$$X^{\beth} \ni v \mapsto \log |I|(v) \doteq -v(I).$$

Clearly, for every ideal I the function  $\log |I|$  is monotone decreasing (with respect to the natural partial order on  $X^{\Box}$ ). Moreover:

$$\log|I \cdot J| = \log|I| + \log|J|, \quad \log|I + J| = \max\{\log|I|, \log|J|\}.$$
 (1.4.1)

**Definition 1.4.2.** The set of functions  $\psi: X^{\Box} \to \mathbb{R}$ , of the form

$$v \mapsto \frac{1}{m} \max\left\{ \log |I_i|(v) + k_i \right\}$$
(1.4.2)

for  $I_1, \ldots, I_N$  ideals, and integers  $m, k_1, \ldots, k_N \in \mathbb{Z}$ , is denoted  $\mathrm{PL}^+(X^{\beth})$ , and the Q-vector space it generates  $\mathrm{PL}(X^{\beth}) \subseteq \mathrm{C}^0(X^{\beth}, \mathbb{R})$ .

An element of  $\operatorname{PL}(X^{\beth})$  will be called a piecewise linear function, we denote  $\operatorname{PL}_{\mathbb{R}}(X^{\beth}) \doteq \operatorname{PL}(X^{\beth}) \otimes_{\mathbb{Q}} \mathbb{R}$ .

**Lemma 1.4.3.**  $PL^+(X^{\beth})$  separates points.

*Proof.* If  $v, w \in X^{\beth}$  are distinct, then there exists  $I \in \mathscr{I}$  such that  $v(I) \neq w(I)$ , with no loss of generality we can assume w(I) < v(I), and thus taking

$$\ell \in ] w(I), v(I) [ \bigcap \mathbb{Q},$$

and  $\varphi \doteq \max\{\log|I|, -\ell\} \in \operatorname{PL}(X^{\beth})$ , we have  $\varphi(v) = -\ell < -w(I) = \varphi(w)$ .

**Proposition 1.4.4.**  $PL(X^{\beth})$  is dense in  $C^0(X^{\beth}, \mathbb{R})$ 

*Proof.* Since PL is a Q-linear subspace of  $C^0$  stable by max, containing the (Q-) constants, and separating points, the result follows from the lattice version of the Stone-Weierstrass Theorem.

# 2. Semivaluations on a complex space X

In this section we study  $X^{\beth}$  attached to a compact analytic variety X. From now on X will always denote such a space.

For a complete reference on the more standard algebraic setting see [BJ22], most of the results proved here, are direct analogues of results found there.

We recall the definition of  $X^{\Box}$ :

$$X^{\beth} = \left\{ \begin{array}{c} v \colon \mathscr{I}_X \to [0, +\infty] \\ v \colon \mathscr{I}_X \to [0, +\infty] \end{array} \middle| \begin{array}{c} v(I \cdot J) = v(I) + v(J) \\ v(I + J) = \min\{v(I), v(J)\} \\ v(\mathcal{O}_X) = 0 \& v(0_X) = +\infty \end{array} \right\},$$

where  $\mathscr{I}_X$  denotes the set of ideals locally of finite type, that for a complex space coincides with the set coherent ideals by Oka's theorem.

We call an element,  $v \in X^{\exists}$ , a *semivaluation*. For  $D \subseteq X$  an effective divisor, and  $v \in X^{\exists}$  we set:

$$v(D) \doteq v\left(\mathcal{O}_X(-D)\right).$$

# 2.1. Support and center of a semivaluation.

**Lemma 2.1.1.** Let  $v \in X^{\beth}$  be a semivaluation, then there exist unique coherent ideals  $I_s(v), I_c(v)$  that satisfy

$$v(I_s) = \infty, \quad and \quad v(I_c) > 0, \tag{2.1.1}$$

and are maximal with this property. Moreover,  $I_s(v)$  and  $I_c(v)$  are prime ideals.

*Proof.* Let  $S \subseteq \mathscr{I}_X$  be the set of ideals on which v is infinite, C the set on which v is positive, and take:

$$I_s \doteq \sum_{J \in S} J$$
, and  $I_c \doteq \sum_{J \in C} J$ .

By the strong noertherian property,  $I_s$  and  $I_c$  are coherent, and by construction satisfy (2.1.1).

For primality, if J an K are ideals on X, such that  $J \cdot K \subseteq I_c(v)$ , then  $0 < v(I_c) \le v(J \cdot K) = v(J) + v(K) \implies$  either 0 < v(J) or 0 < v(K), and hence either  $J \subseteq I_c$  or  $K \subseteq I_c$ , which implies that  $I_c$  is prime. We proceed similarly for  $I_s$ .

**Definition 2.1.2.** Let  $v \in X^{\exists}$ , we denote by  $S_X(v) = S(v)$ , the support of v, the subvariety of X given by  $I_s$ . In the same fashion, we denote by  $Z_X(v) = Z(v)$ , the central variety of v, the subvariety attached to  $I_c$ .

**Remark 2.1.3.** Since  $I_s \subseteq I_c$  it follows that  $Z(v) \subseteq S(v)$ .

Moreover, for any  $p \in X$  the global ideal  $m_p$  is maximal, and thus

$$Z_X(v) = \{p\} \iff v(m_p) > 0, \quad and \quad S_X(v) = \{p\} \iff v(m_p) = \infty.$$

11

It is easy to see that the support and the center are well-behaved under inclusions.

If v is finite valued on nonzero ideals, i.e.  $S_X(v) = X$ , then we will say that v is *valuation* on X. We will denote the set of valuations by  $X^{\text{val}}$ . The support thus give us the following decomposition

$$X^{\beth} = \bigsqcup_{Y \subseteq X} Y^{\text{val}}$$
(2.1.2)

where Y ranges over all the subvarieties of X.

We can think the set of valuations centered at  $\{p\}$  in terms of the Example 1.3.6:

Example 2.1.4. The assignment

$$(X_p)^{\beth} \ni v \mapsto v_p \in X^{\beth}$$

induces a bijection from the set of local semivaluations centered at p to the set of global semivaluations of X centered at p.

Indeed, by Remark 2.1.3 it is clear that the mapping sends semivaluations centered at p to semivaluations centered at p.

On the other hand, if  $Z_X(v) = \{p\}, f \in \mathcal{O}_{X,p}$ , and  $I_f$  the local ideal generated by f. We denote

$$I_k \doteq I_f + m_p^k$$

and observe:

(1)  $I_k$  can be extended to a global ideal, just by triviality outside p.

(2)  $I_{k+1} = I_k + m_p^{k+1} \subseteq I_k$ .

We define  $\nu(f)$  as the decreasing limit:

$$\nu(f) \doteq \lim_{k \to \infty} v(I_k) = \lim_{k \to \infty} \min\{v(I_{k-1}), k \cdot v(m_p)\} \in [0, +\infty].$$

It is easy to see that  $\nu$  is a valuation and that  $\nu_p = v$ .

In the next section we will use the above example to reconstruct  $X^{\beth}$  from  $(X_p)^{\square}$  for every p, whenever X a smooth analytic curve.

**Remark 2.1.5.** A locally ringed space  $(X, \mathcal{O}_X)$  satisfies the strong noetherian property if locally every increasing chain of  $\mathcal{O}_X$ -ideals, locally of finite type, is locally stationary.

This is the property needed to defined the center and support of a semivaluation, since Lemma 2.1.1 holds in this case.

**Proposition 2.1.6.** If  $\pi: Y \to X$  is a bimeromorphic morphism, the map

$$\pi^{\beth} \colon Y^{\beth} \to X^{\beth}$$

induces a bijection  $Y^{\text{val}} \simeq X^{\text{val}}$ .

*Proof.* We first observe that  $\pi^{\neg}$  maps valuations to valuations: if  $v \in Y^{\neg}$  is finite valued on the set of non-zero ideals of Y, then we need to check that, for a non-zero ideal I of  $X, \pi^{-1}(I)$  is not zero. Let  $U \subseteq X$  be an open set such that I is not zero, then  $\pi^{-1}(U)$  is an open set bimeromorphic to U, and

$$\pi^{-1}I(U) = \{0\} \iff \pi(U) \subseteq Z_I,$$

where  $Z_I$  is the zero locus of I, that has strictly positive codimension on U, therefore  $\pi^{-1}I \neq 0$ . Thus  $\pi^{\neg}(v)(I) < +\infty$ , for I a non-zero ideal on X.

Let's prove the that  $\pi^{\Box}$  induces the desired bijection.

First suppose that there exists an anti-effective divisor  $E \subseteq Y$  that is  $\pi$ -ample. Then for every ideal of  $Y, J \in \mathscr{I}_Y$ , choosing  $m_J \in \mathbb{N}$  sufficiently large,  $J \cdot \mathcal{O}_Y(E)^{m_J}$  is  $\pi$ -globally generated, that is

$$\pi^* I_J = J \cdot \mathcal{O}_Y(E)^{m_J},\tag{2.1.3}$$

for some ideal of  $X, I_J \in \mathscr{I}_X$ . Since  $\mathcal{O}_Y(E)$  is also  $\pi$ -globally generated we can also find  $K \in \mathscr{I}$  such that

$$\pi^* K = \mathcal{O}_Y(E).$$

Hence, given  $v_X \in X^{\text{val}}$  the function:

$$\mathscr{I}_Y \ni J \mapsto v_X \left( I_J \right) - m_J v_X \left( K \right)^4 \tag{2.1.4}$$

is a valuation on Y, which we will denote by  $v_Y$ , such that  $\pi^{\beth}(v_Y) = v_X$ . If, for  $v, w \in Y^{\text{val}}, \pi^{\beth}(v) = \pi^{\beth}(w)$ , then for every J ideal on Y:

$$\begin{aligned} v(J) &= v\left(J \cdot \mathcal{O}_Y(E)^{m_J}\right) - m_J v\left(\mathcal{O}_Y(E)\right) \\ &= v(\pi^* I_J) - m_J v(\pi^* K) \\ &= \pi^{\neg}(v)(I_J) - m_J \pi^{\neg}(v)(K) \\ &= \pi^{\neg}(w)(I_J) - m_J \pi^{\neg}(w)(K) \\ &= w(\pi^* I_J) - m_J w(\pi^* K) \\ &= w\left(J \cdot \mathcal{O}_Y(E)^{m_J}\right) - m_J w\left(\mathcal{O}_Y(E)\right) = w(J), \end{aligned}$$

which implies that v = w.

If Y does not admit a divisor E as above, by Hironaka, we can find Y' a smooth complex analytic space:



such that  $\mu$  is a sequence of blow-ups of smooth center, and  $\nu$  is a bimeromorphic morphism.

Thus taking  $E_{\mu}$  the exceptional divisors of  $\mu$ , we have -by the Negativity Lemma of [KM98, Lemma 3.39]- that  $E_{\mu}$  will be anti-effective, and relatively ample, giving the bijection



which implies that  $\pi^{\exists}|_{Y^{\text{val}}}$  is surjective on  $X^{\text{val}}$ , and that  $\nu^{\exists}|_{(Y')^{\text{val}}}$  is injective. In turn, the same argument gives the surjection

$$\nu^{\beth} \colon (Y')^{\mathrm{val}} \twoheadrightarrow Y^{\mathrm{val}}.$$

Since  $\mu^{\neg}$  and  $\nu^{\neg}$  induce bijections on the set of valuations, so does  $\pi^{\neg}$ .

<sup>&</sup>lt;sup>4</sup>Observe that we can take the difference because  $v_X$  is a finite valued.

2.2. Integral closure of an ideal and PL functions. In this section we can be in a slightly more general setting of either complex analytic spaces or excellent schemes of equi-characteristic 0. The two examples, we have in mind are complex analytic spaces, and the the scheme Spec  $\mathcal{O}_{\mathbb{C}^n,0}$ .

For simplicity of the exposure from here on X will be of normal singularities, see [GR12] for a complete reference on normal complex spaces.

**Definition 2.2.1.** Let  $I \in \mathscr{I}_X$ , we consider  $\overline{I}$  the integral closure of I to be the ideal given locally by all the elements  $f \in \mathcal{O}_X$  that satisfy a polynomial equation

$$f^d = \sum_{i=0}^{d-1} a_i f^i \tag{2.2.1}$$

for  $a_i \in I$ .

It turns out that for complex analytic spaces, or excellent schemes as above, the ideal  $\overline{I}$  is a coherent ideal, and hence  $\overline{I} \in \mathscr{I}_X$ .

This follows indeed from the following geometric description of the integral closure of an ideal, that will be useful later, given by the following results:

**Proposition 2.2.2.** Let  $I \in \mathscr{I}$ ,  $\nu: Y \to X$  the normalized blow-up of X along I, and  $E \subseteq Y$  the exceptional divisor. We then have that  $\nu_*(\mathcal{O}_Y(-E)) = \overline{I}$ .

Proof. See [Laz17, Proposition 9.6.6]

**Corollary 2.2.3.** Let  $\mu: Y \to X$  be a projective modification of  $X, D \subset Y$  an effective divisor such that

$$\mathcal{O}_Y(-D) = I \cdot \mathcal{O}_Y \tag{2.2.2}$$

for some I ideal of  $\mathcal{O}_X$ . Then  $\mu_*(\mathcal{O}_Y(-D)) = \overline{I}$ .

*Proof.* By Equation (2.2.2)  $\mu$  factors through the normalized blow-up of X along I,  $\widetilde{\text{Bl}_I X} \to X$ , and the result follows from Proposition 2.2.2.

**Lemma 2.2.4.** If I is a coherent ideal then the associated function satisfies

$$\log|I| = \log|\overline{I}| \tag{2.2.3}$$

on  $X^{\text{val}}$ .

*Proof.* Let  $k \in \mathbb{N}$  such that  $I \cdot \overline{I^k} = \overline{I} \cdot \overline{I^k} {}^5$ ,

$$\log|I| + \log|\overline{I^{k}}| = \log|I \cdot \overline{I^{k}}| = \log|\overline{I} \cdot \overline{I^{k}}| = \log|\overline{I}| + \log|\overline{I^{k}}|$$
$$\implies \log|I| = \log|\overline{I}|.$$

The converse also holds, the valuative criterion of integral closedness gives us that if  $\log |I| = \log |J|$ , then  $\overline{I} = \overline{J}$ . Later, in Section 2.6, we will see a more general statement that will imply it.

<sup>&</sup>lt;sup>5</sup>See [Bou18, Remark 8.7]

2.3. Divisorial and monomial valuations. The trivial character described before, will be denoted  $v_{\text{triv}}$  on  $X^{\beth}$ , and called the *trivial valuation*:

$$v_{\text{triv}}(I) = 0$$
, for every non-zero ideal I.

More interestingly, to each irreducible divisor F, of a normal analytic variety  $Y \xrightarrow{\mu} X$  bimeromorphic to X, we can associate a valuation on Y –and hence on X by Proposition 2.1.6– given by:

$$\mathscr{I}_Y \ni I \mapsto \mathrm{ord}_F(I),$$

where  $\operatorname{ord}_F$  is given by the following procedure: choose any point  $q \in F$ , consider  $\operatorname{ord}_{F_q} \in (Y_q)^{\square}$  the order of vanishing along the germ of F at q, and denote by  $\operatorname{ord}_F \in Y^{\operatorname{val}}$  the induced global valuation of Example 1.3.6:

$$\operatorname{ord}_F \doteq (\operatorname{ord}_{F_q})_q$$

It is classical that this construction does not depend on  $q \in F$ . More details will be given in the discussion below on monomial valuations, see Proposition 2.3.4.

Equivalently,  $\operatorname{ord}_F(I) = k$  if and only if we can find a decomposition

$$I = \mathcal{O}_Y(-kF) \cdot J$$

where  $F \not\subseteq Z_J$ , the zero set of J.

**Definition 2.3.1.** We denote by  $X^{\text{div}}$  the set of valuations of the form:

 $r \cdot \operatorname{ord}_F \colon \mathscr{I}_X \setminus \{0_X\} \to \mathbb{Q},$ 

for a rational number  $r \in \mathbb{Q}_{\geq 0}$  and F a divisor as above.

We say that an element of  $X^{\text{div}}$  is a divisorial valuation. In particular, the trivial valuation  $v_{\text{triv}}$  is a divisorial valuation.

**Remark 2.3.2.** Divisorial valuations are a birational invariant: if  $f: Y \to X$  is a bimeromorphic morphism, then  $f^{\square}$  maps divisorial valuations to divisorial valuations, moreover the restriction

$$f^{\beth}|_{Y^{\operatorname{div}}} \colon Y^{\operatorname{div}} \to X^{\operatorname{div}}$$

is bijective.

The study of divisorial valuations will be of central importance for the following, as they encode a lot of the geometry of X.

For an analytic curve,  $X^{\neg}$  can be completely described in terms of divisorial valuations, as we can see in the next example:

**Example 2.3.3.** Let X be a smooth analytic curve, and  $v \in X^{\exists}$  a semivaluation on X. The central variety of v,  $Z_X(v)$ , is an irreducible subvariety of X, therefore either Z(v) = X or  $Z(v) = \{p\}$  a point on X. Let's study the two options:

(1) If 
$$Z(v) = X$$
 then  $v = v_{triv}$ .

(2) If 
$$Z(v) = \{p\}$$
, then

 $v = t \cdot \operatorname{ord}_p$ 

for  $t \doteq v(m_p)$ .

Indeed, v being centered at p implies, by Example 2.1.4, that  $v = w_p$  for some local semivaluation, w, centered at p. In addition, in dimension 1, the ring  $\mathcal{O}_{X,p}$  is a discrete valuation ring and thus  $w = t \cdot \operatorname{ord}_p$ . Hence we get  $v = (t \cdot \operatorname{ord}_p)_p = t \cdot \operatorname{ord}_p$ .

Before further studying the divisorial valuations, we will study a slightly more general class of examples of points on  $X^{\beth}$ , the so-called *monomial valuations*.

2.3.1. Monomial valuations and cone complexes. Let (Y, B) be a snc pair over X, i.e. a smooth bimeromorphic model of X together with  $B = \sum_{i \in I} B_i$  a reduced simple normal crossing (snc) divisor. Define, for each subset  $J \subseteq I$ , the intersection  $B_J \doteq \bigcap_{i \in J} B_j$ . The connected components of each  $B_J$  are called the strata of B, and we usually denote a stratum of B, by the letter Z.

We can associate to the pair (Y, B) a cone complex,  $\hat{\Delta}(Y, B)$ , given by the following rule: for each  $J \subseteq I$ , and each connected component, Z, of  $B_J$ , we associate  $\hat{\sigma}_Z \subset \mathbb{R}^I$ , a cone identified with  $(\mathbb{R}_+)^J$ .

We give now a procedure for assigning to each point

$$w \in \hat{\sigma}_Z \subseteq \Delta(Y, B)$$

a valuation on X:

Take any point  $p \in Z$ , and let's suppose for convenience that  $J = \{1, \ldots, k\}$ , we can construct a monomial valuation on  $\mathcal{O}_{X,p}$  with respect to the germs  $B_1, \ldots, B_k$  at p and weights  $w_1, \ldots, w_k$  following [JM12]. This gives us a valuation  $\operatorname{val}_p(w) \in \operatorname{Val}(\mathcal{O}_{X,p}) =$  $(X_p)^{\mathrm{val}}$ . If we are given a coordinate open set  $z: U \to \mathbb{D}^n$  around p, such that in U the divisor  $B_j$  is given by the local equation  $z_j = 0$ , for  $j \in J$ , then the valuation  $val_p(w)$ is given by:

$$\operatorname{val}_p(w)(f) = \min_{\alpha \neq 0} \langle \alpha, w \rangle,$$

for  $f \in \mathcal{O}_{X,p}$ , and  $f = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha} z^{\alpha}$ , in a possibly smaller open set. By some standard algebraic machinery, like in [MN15, JM12],  $\operatorname{val}_p(w)$  does not depend on the coordinates chosen, and only on the divisors. For an analytic proof see Appendix B.

Just like in Example 1.3.6, consider the valuation  $(\operatorname{val}_p(w))_p \in X^{\square}$ . We will now argue that  $(\operatorname{val}_p(w))_p$  does not depend on the point chosen  $p \in Z^6$ , and therefore for  $w \in \hat{\sigma}_Z \subseteq \hat{\Delta}(Y, B)$  we denote:

$$\operatorname{val}(w) \doteq (\operatorname{val}_p(w))_p. \tag{2.3.1}$$

**Proposition 2.3.4.** For any  $w \in \hat{\sigma}_Z$  and  $p \in Z$ , the image in  $X^{\text{val}}$  of  $\text{val}_p(w) \in (X_p)^{\text{val}}$ is independent of the choice of p.

*Proof.* By connectedness of Z, it is enough to show that the statement holds locally near a given  $p \in X$ .

Let  $z: U \to \mathbb{D}^n$  be a local coordinate chart around p such that:

- z(p) = 0;
- B<sub>J</sub> ∩ U = Z ∩ U;
  B<sub>j</sub> ∩ U = (z<sub>j</sub> = 0), for every j ∈ J.

Let  $q \in z^{-1}\left(\mathbb{D}(\frac{1}{3})\right) \cap Z$ , and  $f \in \mathcal{O}_X(U)$ , we'll prove that  $\operatorname{val}_p(w)(f) = \operatorname{val}_q(w)(f)$ . Before going on, we introduce some notation that will be useful. Denote by  $z = (\underline{z}_1, \underline{z}_2)$  in such a way that  $\underline{z}_1 \in \mathbb{C}^k$ , and  $\underline{z}_2 \in \mathbb{C}^{n-k}$ .

Using the coordinate system z to identify U and  $\mathbb{D}^n$ , writing  $q = (\underline{q}_1, \underline{q}_2)$ , we have that  $q_1 = 0$ , and if

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

 $<sup>^{6}</sup>$ Again, the analytic point of view of the monomial valuations explored in Appendix B, give us the local independance of p, but for simplicity we will give here a more elementary approach.

is the expansion of f at 0, then:

$$f(q + z') = \sum c_{\alpha}(q + z')^{\alpha}$$
  
=  $\sum c_{\alpha_1,\alpha_2} (\underline{q}_1 + \underline{z'}_1)^{\alpha_1} (\underline{q}_2 + \underline{z'}_2)^{\alpha_2}$   
=  $\sum c_{\alpha_1,\alpha_2} \underline{z'}_1^{\alpha_1} (\underline{q}_2 + \underline{z'}_2)^{\alpha_2}$   
=  $\sum c_{\alpha_1,\alpha_2} \underline{z'}_1^{\alpha_1} \left[ \sum_j {\alpha_2 \choose j} \underline{q}_2^j \underline{z'}_2^{\alpha_2 - j} \right]$   
=  $\sum \left[ \sum_j c_{\alpha_1,\alpha_2+j} \underline{q}_2^j {\alpha_2+j \choose j} \right] \underline{z'}_1^{\alpha_1} \underline{z'}_2^{\alpha_2},$ 

denoting  $c'_{\alpha_1,\alpha_2} \doteq \sum_j c_{\alpha_1,\alpha_2+j} \underline{q}_2^j {\alpha_2+j \choose j}$ , the exapansion of f around q becomes:

$$f(q+z') = \sum c'_{\alpha_1,\alpha_2} \underline{z'_1}^{\alpha_1} \underline{z'_2}^{\alpha_2},$$

hence we get:  $\operatorname{val}_q(w)(f) = \min_{c'_{\alpha_1,\alpha_2} \neq 0} \langle w, \alpha_1 \rangle$ , and

Now, if  $c'_{\alpha_1,\beta} \neq 0$  for some  $\alpha_1 \in \mathbb{N}^k$  and  $\beta \in \mathbb{N}^{n-k}$ , there exists  $\alpha_2 \in \mathbb{N}^{n-k}$  such that  $c_{\alpha_1,\alpha_2} \neq 0$ . By Equation (2.3.2) we thus get that  $\operatorname{val}_p(w)(f) \leq \operatorname{val}_q(w)(f)$ .

As long as q is sufficiently close to p, close enough to find a coordinate chart around q that contains p with the property that the divisors  $B_1, \ldots, B_k$  are given by the equations  $(z_1 = 0), \ldots, (z_k = 0)$ , we can exchange the role of p and q, and get the equality:

$$\operatorname{val}_p(w)(f) = \operatorname{val}_q(w)(f). \tag{2.3.3}$$

Thus, if we let I be a coherent ideal, there exists an open set U containing p, and local generators  $f_1, \ldots, f_{\rho} \in I(U)$  such that at any point  $q \in U$ , the germs  $(f_1)_q, \ldots, (f_{\rho})_q$  generate  $I_q$ . By Equation 2.3.3, this implies that:

$$\operatorname{val}_p(w)(I_p) = \operatorname{val}_q(w)(I_q),$$

for  $q \in U$ . Therefore, we have just proved that for every I coherent ideal  $(\operatorname{val}_p(w))_p(I)$  is locally independent of p, getting the desired result.

The above construction of monomial valuations gives us an embedding:

val: 
$$\hat{\Delta}(Y, B) \hookrightarrow X^{\beth}$$
.

In fact, if  $w \neq w'$  both lie on  $\hat{\sigma}_Z$ , then there exists an irreducible component of  $B, Z \subseteq B_i$ , such that  $\operatorname{val}(w)(B_i) \neq \operatorname{val}(w')(B_i)$ .

An element of the image of the above mapping is called a *monomial valuation*.

**Remark 2.3.5.** A monomial valuation associated to a rational point on  $\hat{\Delta}(Y, B)$  is a divisorial valuation. As in the algebraic setting, this can be seen using weighted blowups.

2.3.2. Back to divisorial valuations. As stated before, divisorial valuations will be of fundamental importance for our study of the geometry of X. As an example, one is able to prove that  $X^{\text{div}}$  alone can tell homogeneous PL functions apart. To see that, one proves that the homogeneous PL functions are essentially (Q-Cartier) b-divisors over X, and the value on divisorial valuations will completely determine the associated b-divisor. This idea will be further developed in Section 2.6, when we'll prove that  $PL(X^{\Box})$  is isomorphic to a set of  $\mathbb{C}^*$ -equivariant Q-Cartier b-divisors over  $X \times \mathbb{P}^1$ .

Now we will focus our attention to obtain Theorem B.

**Theorem 2.3.6** (Theorem B).  $X^{\text{div}}$  is dense in  $X^{\beth}$ .

Since  $\operatorname{PL}(X^{\beth})$  is dense in  $\operatorname{C}^{0}(X^{\beth}, \mathbb{R})$ ,  $\operatorname{PL}(X^{\beth})$  separates points from closed sets, and hence to prove Theorem 2.3.6, it is enough to show that  $\varphi(X^{\operatorname{div}}) = \{0\} \implies \varphi = 0$ , for every  $\varphi \in \operatorname{PL}(X^{\beth})$ .

Therefore we can restate Theorem 2.3.6, in the following way:

**Statement.** If  $\varphi \in PL(X^{\beth})$  is such that

$$\varphi|_{X^{\mathrm{div}}} = 0$$

then  $\varphi = 0$ .

General idea of the proof of Theorem 2.3.6. In order to prove Theorem 2.3.6, it will be useful to write a PL function as the evaluation function of some ideal.

Let's assume that it is the case, and  $\varphi \in \text{PL}$  is attached to an ideal I, i.e.  $\varphi = \log|I|$ , then if  $\log|I|(v) = 0$  for every  $v \in X^{\text{div}}$ , and if  $\overline{I} \neq \mathcal{O}_X$ , it would be enough to take an irreducible component of the exceptional divisor  $F \subseteq \widehat{\text{Bl}_I X}$ , and consider  $\operatorname{ord}_F \in X^{\text{div}}$ . This would give us  $0 = \log|I|(\operatorname{ord}_F) = -\operatorname{ord}_F(I) \neq 0$ .

Just like in the algebraic trivially valued case of [BJ22], this lead us to consider  $\mathbb{C}^*$ -equivariant models of  $X \times \mathbb{P}^1$ , since we will be able to see  $\mathrm{PL}^+(X^{\Box})$  as the evaluation functions attached to  $\mathbb{C}^*$ -equivariant ideals.

2.4.  $\mathbb{C}^*$ -equivariant non-archimedean space. Recall that  $\mathfrak{a}$  is a flag ideal of  $X \times \mathbb{P}^1$  if it is  $\mathbb{C}^*$ -equivariant coherent ideal of  $X \times \mathbb{P}^1$  whose support is contained in  $X \times \{0\}$ .

Let  $\mathcal{F}$  be the set of fractional flag ideals of  $X \times \mathbb{P}^1$ , and consider the following subset of  $(X \times \mathbb{P}^1)^{\beth}$ :

$$(X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*} \doteq \left\{ \begin{array}{c} v \colon \mathcal{F} \to \mathbb{R} \\ v \colon \mathcal{F} \to \mathbb{R} \end{array} \middle| \begin{array}{c} v(\mathfrak{a} \cdot \mathfrak{b}) = v(\mathfrak{a}) + v(\mathfrak{b}) \\ v(\mathfrak{a} + \mathfrak{b}) = \min\{v(\mathfrak{a}), v(\mathfrak{b})\} \\ v(t) = 1 \quad \& \quad v(\mathcal{O}_{X \times \mathbb{P}^1}) = 0 \end{array} \right\}$$
(2.4.1)

together with a continuous map,  $\sigma$ , the Gauss extension map

$$\sigma \colon X^{\beth} \to (X \times \mathbb{P}^1)_{\mathbb{C}^*}^{\beth} \subseteq (X \times \mathbb{P}^1)^{\beth}$$

where  $\sigma(v) \colon \mathcal{F} \to \mathbb{R}$ , the Gauss extension of v, is given by

$$\sigma(v)(\mathfrak{a}) = \sigma(v)(\sum_{\lambda \in \mathbb{Z}} a_{\lambda} t^{\lambda}) \doteq \min_{\lambda} \{v(a_{\lambda}) + \lambda\} \in \mathbb{R}.$$
 (2.4.2)

**Remark 2.4.1.** As said before, we can see  $(X \times \mathbb{P}^1)_{\mathbb{C}^*}^{\neg}$  as a subset of  $(X \times \mathbb{P}^1)^{\neg}$ . Given  $v \in (X \times \mathbb{P}^1)_{\mathbb{C}^*}^{\neg}$ , we can extend it to the set of all ideals  $\mathscr{I}_{X \times \mathbb{P}^1}$ , by setting

$$v(I) \doteq \lim_{k \to \infty} v\left((t^k) + \sum_{\lambda \in \mathbb{C}^*} \lambda^* I\right)$$

for I an ideal in  $X \times \mathbb{P}^1$ .

Lemma 2.4.2. The Gauss extension,

$$\sigma\colon X^{\beth}\to (X\times\mathbb{P}^1)^{\beth}_{\mathbb{C}^*},$$

is a bijection.

Proof. In fact, taking  $r: (X \times \mathbb{P}^1)^{\neg}_{\mathbb{C}^*} \to X^{\neg}$  to be the restriction map  $r(v): \mathscr{I}_X \ni I \mapsto v (I \cdot \mathcal{O}_{X \times \mathbb{P}^1})$ 

we get the inverse of  $\sigma$ . It is clear that, as defined, r(v) is a semivaluation on X. So we are left to checking that for  $\sigma(r(v)) = v \in (X \times \mathbb{P}^1)^{\neg}_{\mathbb{C}^*}$ , and  $r(\sigma(v)) = v \in X^{\neg}$ , which follows from:

$$\sigma(r(v))(\mathfrak{a}) = \min \{r(v)(\mathfrak{a}_{\lambda}) + \lambda\}$$
  
= min { $v(\mathfrak{a}_{\lambda} \cdot \mathcal{O}_{X \times \mathbb{P}^{1}}) + \lambda$ }  
= min { $v(\mathfrak{a}_{\lambda} \cdot \mathcal{O}_{X \times \mathbb{P}^{1}} \cdot (t^{\lambda}))$ }  
=  $v\left(\sum_{\lambda} \mathfrak{a}_{\lambda} \cdot \mathcal{O}_{X \times \mathbb{P}^{1}} \cdot (t^{\lambda})\right),$   
=  $v(\mathfrak{a})$ 

and

$$r(\sigma(v))(I) = \sigma(v)(I \cdot \mathcal{O}_{X \times \mathbb{P}^1})$$
  
=  $\sigma(v)\left(\sum_{\lambda \ge 0} I \cdot t^{\lambda}\right) = \min\{v(I) + \lambda\}$   
=  $v(I).$ 

For simplicity sometimes we identify  $X^{\beth}$  and  $(X \times \mathbb{P}^1)_{\mathbb{C}^*}^{\beth}$ . The set of PL functions has a nicer description in  $(X \times \mathbb{P}^1)_{\mathbb{C}^*}^{\beth}$ . If we denote

$$\varphi_{\mathfrak{a}} \doteq \log|\mathfrak{a}| \circ \sigma$$

we get the following result:

**Proposition 2.4.3.** The set  $\{\varphi_{\mathfrak{a}} \mid \mathfrak{a} \in \mathcal{F}\}$  Q-generates  $PL(X^{\beth})$ , the space of PL functions of  $X^{\beth}$ . Moreover

$$\mathrm{PL}_{+}(X^{\beth}) = \left\{ \frac{1}{m} \varphi_{\mathfrak{a}} \mid \mathfrak{a} \in \mathcal{F}, m \in \mathbb{Z} \right\}.$$

*Proof.* The proof follows form the description of flag ideals given in the equation (0.0.6).

As we remarked at the end of Section 2.3, this is a step in the right direction in order to prove Theorem 2.3.6. But then two problems rise up:

- (1) The argument at the end of section 2.3 gives us a  $\mathbb{C}^*$ -equivariant divisorial valuation  $v_E \in (X \times \mathbb{P}^1)^{\square}_{\mathbb{C}^*}$ , but a priori we don't know if  $v_E$  comes from a divisorial valuation over X, i.e. the restriction of  $v_E$ ,  $r(v_E)$ , lies in  $X^{\text{div}}$ . Hence, we don't get a contradiction.
- (2) Even though the set  $PL_+$  generates the PL functions, it is not enough to check that Theorem 2.3.6 is true for a  $PL_+$  function. What we need to check is that if  $\varphi_1(\operatorname{ord}_F) = \varphi_2(\operatorname{ord}_F)$  for every  $\operatorname{ord}_F \in X^{\operatorname{div}}$  then  $\varphi_1 = \varphi_2$ .

Section 2.5 will deal with the first problem, and Section 2.6 with the second one.

19

### 2.5. $\mathbb{C}^*$ -equivariant divisorial valuations.

**Definition 2.5.1** (Test configuration). We define a test configuration for X as the data of

- a normal compact Kähler space  $\mathcal{X}$ ;
- $a \mathbb{C}^*$ -action on  $\mathcal{X}$ ;
- a  $\mathbb{C}^*$ -equivariant flat morphism  $\pi \colon \mathcal{X} \to \mathbb{P}^1$ ;
- a  $\mathbb{C}^*$ -equivariant biholomorphism

$$\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\}),$$

where  $\mathcal{X}_0 \doteq \pi^{-1}(0)$ .

Moreover, if  $\mathcal{X}$ , and  $\mathcal{X}'$  are test configuration,  $\mathcal{X}$  dominates  $\mathcal{X}'$  if the bimeromorphic map

$$\mathcal{X} \dashrightarrow X \times \mathbb{P}^1 \dashrightarrow \mathcal{X}'$$

extends to a  $\mathbb{C}^*$ -equivariant holomorphic map, which we say is a morphism of test configurations. When  $\mathcal{X}$  dominates  $X \times \mathbb{P}^1$ , we say that  $\mathcal{X}$  is dominating.

**Remark 2.5.2.** Test configurations of a given normal analytic variety X define a directed system. By the equivariant version of Hironaka's theorem, the set of test configurations that are projective over  $X \times \mathbb{P}^1$ , and of snc central fiber, are cofinal in all test configurations.

Unless otherwise stated, we will consider always such test configurations.

An important class of examples of test configuration are given by flag ideals:

**Example 2.5.3.** Let  $\mathfrak{a}$  be a flag ideal on  $X \times \mathbb{P}^1$ , then

$$\mathcal{X} \doteq \mathrm{Bl}_{\mathfrak{a}}(X \times \mathbb{P}^1)$$

is a test configuration of X that dominates  $X \times \mathbb{P}^1$ .

For more examples see [DR17b, Example 2.11].

**Definition 2.5.4** ( $\mathbb{C}^*$ -invariant divisorial valuations). Let  $\mathcal{X}$  be a test configuration of X that dominates  $X \times \mathbb{P}^1$ , and  $\mathcal{X}_0 = \sum b_E E$  its decomposition into irreducible components. We can associate to E an element,  $v_E$ , of  $(X \times \mathbb{P}^1)^{\text{div}} \cap (X \times \mathbb{P}^1)^{\neg}_{\mathbb{C}^*}$  by the formula:

$$v_E(\mathfrak{a}) \doteq \frac{1}{b_E} \operatorname{ord}_E(\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}}).$$
 (2.5.1)

We denote the set of such valuations, for all test configurations  $\mathcal{X}$ , and all irreducible  $E \subseteq \mathcal{X}_0$  irreducible components of the central fibers, by  $(X \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*} \subseteq (X \times \mathbb{P}^1)^{\neg}_{\mathbb{C}^*}$ .

Now, we focus our attention to  $(X \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*}$ , and prove the following theorem:

**Theorem 2.5.5.** Let  $r: (X \times \mathbb{P}^1)^{\square} \to X^{\square}$  be the restriction map of Lemma 2.4.2, we then have

$$r((X \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*}) = X^{\operatorname{div}}.$$

This is known in the algebraic case, see [BHJ17, Theorem 4.6]. In that context, the proof relies on some valuative machinery that we don't have at our disposal here. More specifically, when X is a proper algebraic variety over  $\mathbb{C}$ ,  $X^{\beth}$  corresponds to the the Berkovich analytification, whose points are (semi)valuations on the field of functions of (a subvariety of) X. Hence, we can associate to it invariants such as the rational rank, and the transcendental degree, that characterize the divisorial valuations completely.

To prove Theorem 2.5.5 in our transcendental setting we will reduce to the algebraic setting. In order to do that we use Proposition 2.5.10, which states that the order of

vanishing along a smooth divisor can be computed locally. Hence it will be enough to consider "germs" of manifolds and divisors, that is to consider at any  $p \in X$ , the scheme  $X_p = \text{Spec } \mathcal{O}_{X,p}$ , and do local computations.

A useful tool in the following will be the following 'GAGA'/base change theorem.

**Theorem 2.5.6.** Given X an analytic variety, and  $p \in X$  a point. There is an equivalence between the category of projective  $\mathcal{O}_{X,p}$ -schemes, and that of analytic spaces which are projective over the germ of X at p.

Sketch of correspondence. Let's first build the functor on the objects.

Given a projective morphism  $Y \xrightarrow{\pi} U$ ,  $U \subseteq X$  an open set containing p, there exists an embedding  $Y \xrightarrow{j} U \times \mathbb{P}^N_{\mathbb{C}}$  such that



commutes. This means that we can find a finite number of homogeneous polynomials  $f_1, \ldots, f_k \in \mathcal{O}_X(V)[t_1, \ldots, t_{N+1}]$  that cut-out  $Y|_V \doteq \pi^{-1}(V)$ , for some V open neighborhood of p. Taking the germ of the coefficient of  $f_i$  at p we get  $f_1, \ldots, f_k \in \mathcal{O}_{X,p}[t_1, \ldots, t_{N+1}]$ , which defines a subvariety  $Y_p$  of  $\mathbb{P}^N_{\mathcal{O}_{X,p}} \cong \operatorname{Spec} \mathcal{O}_{X,p} \times_{\operatorname{Spec} \mathbb{C}} \mathbb{P}^N_{\mathbb{C}}$  and hence we get a projective morphism  $\pi_p \colon Y_p \to \operatorname{Spec} \mathcal{O}_{X,p}$  given by the diagram



where j is the inclusion.

To analytify a projective morphism over Spec  $\mathcal{O}_{X,p}$  the strategy is the same, see [JM14]. It is clear how the correspondence of the objects induce a correspondence on the morphisms. For more details see and [Bin76].

Apart from the usual correspondence of sheaves, one important property of this 'GAGA' theorem is the following dimension compatibility result:

**Proposition 2.5.7.** Let  $U \subseteq X$  be an open set, and let  $p \in U \subseteq X$ , Z a complex analytic space that is projective over U, and  $q \in Z$  in the fiber of p. Then the dimension at q of the germ of Z over p is equal to the dimension of Z.

Proof. See [Bin76, Theorem 2.8].

Given a projective morphism  $\varphi \colon Y \to X$  we say the *localization of*  $\varphi$  *at* p is its isomorphism class on the category of the projective morphisms over the germ of X at p.

*Birational Geometry intermezzo.* Before proving Theorem 2.5.5, we recall some basic facts of Birational Geometry.

**Remark 2.5.8.** Let  $v \in X^{\text{div}}$  be a divisorial valuation, and  $\mu: X' \to X$  a bimeromorphic morphism such that  $Z \doteq Z_{X'}(v) \subseteq X'$  is a (irreducible) divisor, then  $\operatorname{ord}_Z = v$ .

Indeed, if  $F \subseteq Y \xrightarrow{\pi} X$  is chosen such that  $v = \operatorname{ord}_F$ , then it is enough to choose a bimeromorphic model Y' that dominates Y and X', together with an irreducible divisor,  $F' \subseteq Y'$ , making the diagram,

$$Z \subseteq X' \xleftarrow{\nu} F' \subseteq Y'$$

$$\downarrow^{\mu} \qquad \qquad \downarrow^{\nu'}$$

$$X \xleftarrow{\pi} F \subseteq Y,$$

$$(2.5.2)$$

commute, where  $F' \doteq \nu^{-1}(Z)$  is the strict transform of Z by  $\nu$ . Then  $\operatorname{ord}_E = \operatorname{ord}_Z$ , and  $\operatorname{ord}_E = \operatorname{ord}_F$ , in particular  $\operatorname{ord}_F = \operatorname{ord}_Z$ .

The next Lemma is a version of Zariski's Lemma, found on [Art86, Appendix: Prime Divisors, pg. 229], that will be important for the following.

**Lemma 2.5.9.** Let X be an integral scheme, and v a divisorial valuation of X, then after blowing-up a finite number of times the center of v, c(v), the latter will be the generic point of a divisor.

For more details see [Art86].

Back at the discussion of Theorem 2.5.5. Let's recollect the discussion on Section 2.3, when  $F \subseteq X$  is a prime smooth divisor on X, and p a point in F, then the valuation  $\operatorname{ord}_F \in X^{\operatorname{div}}$  is given by the following procedure:

- (1) Consider the valuation  $\operatorname{ord}_{F_p} \in (X_p)^{\operatorname{val}}$  given by the germ of F at p
- (2) Then define

$$\operatorname{ord}_F(I) \doteq \operatorname{ord}_{F_q}(I_p)$$

where  $I_p$  denotes the germ of I at p. We saw that this definition does not depend on the point  $p \in F$ .

More generally:

**Proposition 2.5.10.** Let  $G \subseteq Y \xrightarrow{\mu} X$  be a (prime smooth) divisor, and  $\mu$  a projective bimeromorphic morphism, consider  $p \in Z_X(\operatorname{ord}_G) = \mu(G)$ . Then, localizing at p we get a (prime smooth) divisor

$$G_p \subseteq Y_p \xrightarrow{\mu_p} X_p$$

and the associated divisorial valuation on  $X_p^{\exists}$  satisfies

$$\operatorname{ord}_{G}(I \cdot \mathcal{O}_{Y}) = \operatorname{ord}_{G_{p}}(I_{p} \cdot \mathcal{O}_{Y_{p}})$$
(2.5.3)

for I a coherent ideal of X, and  $I_p$  the germ of I at p.

*Proof.* Let I be an ideal on X and  $k \doteq \operatorname{ord}_G(I)$ , write:

$$I \cdot \mathcal{O}_Y = \mathcal{O}_Y(-kG) \cdot J,$$

with J an ideal such that  $G \not\subseteq Z_J$ . Localizing at p we get:

$$I_p \cdot \mathcal{O}_{Y_p} = \mathcal{O}_{Y_p}(-kG_p) \cdot J_p.$$

By primality of G we have that  $G_p$  is prime and smooth, in particular  $G_p \nsubseteq Z_{J_p}$ . Getting

$$\operatorname{ord}_{G_p}(I_p \cdot \mathcal{O}_{Y_p}) = k$$

In this  $\mathbb{C}^*$ -equivariant setting we also have an analogue statement as of Remark 2.3.2, that is of key importance for Theorem 2.5.5.

**Proposition 2.5.11.** Let  $f: Y \to X$  be a bimeromorphic morphism, then the morphism  $F \doteq (f, id) \colon Y \times \mathbb{P}^1 \to X \times \mathbb{P}^1$  induces a bijection:

$$F^{\beth}|_{(Y \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*}} \colon (Y \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*} \to (X \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*}.$$

By Remark 2.3.2,  $F^{\Box}$  is a bijection between  $(Y \times \mathbb{P}^1)^{\text{div}}$  and  $(X \times \mathbb{P}^1)^{\text{div}}$ , but since the valuations on  $(X \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*}$  (or  $(Y \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*}$ ) are attached to divisors corresponding to ireducible components of  $\mathbb{C}^*$ -equivariant degenerations of X (or Y resp.), we need to check that, for  $v_E \in (Y \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$ , the divisorial valuation  $F^{\beth}(v_E)$  can be obtained from an irreducible component of the central fiber of a test configuration of X.

Proof of Proposition 2.5.11. Let's start proving that  $F^{\beth}$  maps  $(Y \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*}$  to  $(X \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*}$ .

If  $\mathcal{Y}$  is a test configuration for Y, dominating  $Y \times \mathbb{P}^1$ , and  $E \in \mathrm{VCar}(\mathcal{Y})$  is a prime smooth vertical divisor, then we'll show that  $F^{\square}(v_E) \in (X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$ . That is, that there exists a test configuration  $\mathcal{X}$  for X, together with an irreducible divisor  $D \in \mathrm{VCar}(\mathcal{X})$ and a  $\mathbb{C}^*$ -equivariant birational map  $\mu_T \colon \mathcal{Y} \to \mathcal{X}$ :



such that D and E generate the same divisorial valuation on  $X \times \mathbb{P}^1$ .

To prove this we first observe that  $v \doteq F^{\exists}(v_E)$  is a valuation on  $X \times \mathbb{P}^1$ , and thus, denoting  $X \times \mathbb{P}^1$  by  $\mathcal{X}^1$ , the central variety  $Z_1 \doteq Z(v, \mathcal{X}^1)$  is well defined and a  $\mathbb{C}^*$ invariant irreducible set supported on the central fiber  $X \times \{0\}$ , given by the zeroes of  $(\mu_1)_*\mathcal{O}_{\mathcal{Y}}(-E)$ . Therefore, the blow-up  $\mathcal{X}^2 \doteq \operatorname{Bl}_{Z_1} \mathcal{X}^1$  is a test-configuration for X, and the central variety,  $Z_2$ , of v on  $\mathcal{X}^2$  is  $\mathbb{C}^*$ -invariant, and supported on the central fiber. Inductively, the blow-up  $b_{k+1} \colon \mathcal{X}^{k+1} \to \mathcal{X}^k$  of  $\mathcal{X}^k$  along  $Z_k$ , is a test configuration and

the center,  $Z_{k+1}$ , of v in  $\mathcal{X}^{k+1}$  is  $\mathbb{C}^*$ -invariant and supported on central fiber:

$$E \subseteq \mathcal{Y}$$

$$\downarrow^{\mu_{k}}$$

$$Z_{k} \subseteq \mathcal{X}^{k} \longleftrightarrow_{b_{k+1}} \mathcal{X}^{k+1} \supseteq Z_{k+1} = \overline{\mu_{k+1}(E)},$$

where  $\mu_{k+1}$  is the bimeromorphic map defined by  $\mu_k$  and  $b_{k+1}$ .

In the algebraic case, by a Lemma of Zariski after blowing up the center of the divisorial valuation a finite number of times we get that  $Z(v, \mathcal{X}^k)$  is a divisor. But in our nonalgebraic context, Zariski's result does not a priori apply. The strategy of our proof will be to localize at a point  $p \in Z_1 \subseteq X \times \mathbb{P}^1$ , use the version of Zariski's lemma given in Lemma 2.5.9, that applies in this local case, to get that, for some  $k \gg 0$  sufficiently big,  $(Z_k)_p$  is a divisor. By Proposition 2.5.7 this implies that  $Z_k$  is a divisor, and thus by Remark 2.5.8 we are done.

Taking  $p \in Z_1 \subseteq X \times \mathbb{P}^1$ , and localizing at p we get:



The valuation  $v_{E_p}$  is a divisorial valuation on the  $\mathcal{O}_{X,p}$ -scheme  $(\mathcal{X}^i)_p$  whose (scheme theoretic) center is the generic point of  $(Z_i)_p$ . Since  $(\mathcal{X}^i)_p = \text{Bl}_{(Z_{i-1})_p}(\mathcal{X}^{i-1})_p$ , by Lemma 2.5.9 after a finite number of steps  $(Z_k)_p$  becomes a divisor. By irreducibility of  $Z_k$  and Proposition 2.5.7,  $Z_k$  is a -global- divisor of  $\mathcal{X}^k$ .

The map  $F^{\square}|_{(Y \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*}}$  is injective by Proposition 2.1.6, and it is easy to see that it is surjective.

On to the proof of Theorem 2.5.5:

Proof of Theorem 2.5.5. Let's start proving that  $X^{\text{div}} \subseteq r((X \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*})$ , that is for every  $F \subseteq Y \xrightarrow{\mu} X$ , irreducible smooth divisor on bimeromorphic a model of X,

$$\operatorname{ord}_F \in r((X \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*})$$

Let  $\mathcal{Y}$  be the deformation to the normal cone of  $F \subseteq Y$ , that is the blow-up of  $F \times \{0\}$  in  $Y \times \mathbb{P}^1$ , with exceptional divisor  $E \subseteq \mathcal{Y} \xrightarrow{\mu} Y \times \mathbb{P}^1$ , the irreducible divisor corresponding to the blow-up of Y along F, cf. [Ful98, Chapter 5]. Localizing at  $p \in F$ :

$$F_p \subseteq Y_p$$
 &  $E_p \subseteq \mathcal{Y}_p = \operatorname{Bl}_{F_p \times \{0\}}(Y_p \times_{\operatorname{Spec} \mathbb{C}} \mathbb{P}^1).$ 

Now, applying Proposition 2.5.10 we get that for any ideal *I*:

$$\operatorname{ord}_{F}(I) = \operatorname{ord}_{F_{p}}(I_{p}) = r(v_{E_{p}})(I_{p}) = v_{E_{p}}\left(I_{p} \cdot \mathcal{O}_{\mathcal{Y}_{p}}\right) = v_{E}(I \cdot \mathcal{O}_{\mathcal{Y}}),$$
(2.5.5)

where the second equality is given by  $[BHJ17, Theorem 4.8]^7$ , and thus

$$\operatorname{ord}_F = r(v_E) \in r(\mathcal{Y}_{\mathbb{C}^*}).$$

By Proposition 2.5.11 we have stablished that  $X^{\text{div}} \subseteq r((X \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*})$ .

To prove that  $X^{\operatorname{div}} \supseteq r((X \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*})$ , the strategy will be the same.

Let  $\mathcal{X}$  be a test configuration that dominates  $X \times \mathbb{P}^1$ , and  $E \subseteq \mathcal{X}_0$  an irreducible component. We then have:

<sup>&</sup>lt;sup>7</sup>The set up there is for a scheme of finite type over a field of characteristic zero, but the same arguments apply.

where  $Z_0 \doteq Z(r(v_E), X)$  is the central variety of  $r(v_E)$  on X, and  $\mathcal{Y}^1$  is dominates  $Y_1$ and  $\mathcal{X}$ , and  $f_1$  is a bimeromorphism such that the strict transform  $E_1 = f_1^{-1}E$  is an irreducible smooth divisor. We can define then  $Z_1 \doteq Z(r(v_E), Y_1) = \overline{\rho_1(E_1)} \subseteq Y_1$ .

Localizing at a point  $p \in Z_0$  we get:

and, as before,  $E_p$  defines a divisorial valuation on  $X_p$  –again by the same arguments as in [BHJ17, Theorem 4.8]– and its schematic center in  $X_p$  is the generic point of  $(Z_0)_p$ , similarly the center of  $v_{E_p}$  on  $(Y_1)_p$  is the generic point  $(Z_1)_p$ . Repeating the construction we get:

and

Again by Lemma 2.5.9, it exists  $k \in \mathbb{N}$  such that  $(Z_k)_p \subseteq (Y_k)_p = \operatorname{Bl}_{(Z_{k-1})_p}(Y_{k-1})_p$  is a prime divisor. Proposition 2.5.7, together with the irreducibility of  $Z_k$ , gives us that  $Z_k \subseteq Y_k$  is a prime divisor. Moreover, by construction,  $Z_k$  is the central variety of  $v_E$  on  $Y_k$ , and therefore by Remark 2.5.8:

$$v_{E_p} = \operatorname{ord}_{(Z_k)_p}$$

Thus

$$v_E(I \cdot \mathcal{O}_{\mathcal{X}}) = v_{E_p}(I_p \cdot \mathcal{O}_{\mathcal{X}_p}) = \operatorname{ord}_{(Z_k)_p}(I_p \cdot \mathcal{O}_{(Y_k)_p}) = \operatorname{ord}_{Z_k}(I \cdot \mathcal{O}_{Y_k}),$$
(2.5.8)

getting

$$r(v_E) = \operatorname{ord}_{Z_k} \in X^{\operatorname{div}}.$$

This shows that  $r((X \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*}) \subseteq X^{\operatorname{div}}$ , completing the proof.

# 2.6. PL functions as divisors.

**Definition 2.6.1.** Let  $\mathcal{X}$  be a test configuration of X, we denote by  $VCar(\mathcal{X})$  the finite dimensional Q-vector space given by the  $\mathbb{C}^*$ -invariant Q-Cartier divisors supported on  $\mathcal{X}_0$ . An element of  $VCar(\mathcal{X})$  is called a vertical Q-Cartier divisor on  $\mathcal{X}$ , or simply, vertical divisor of  $\mathcal{X}$ .

Now, since morphisms of test configurations induce linear mappings between the vertical Q-Cartier divisors, we define *vertical Cartier b-divisors*.

**Definition 2.6.2.** Consider the direct system given by  $\langle VCar(\mathcal{X}), \mu^*_{\mathcal{X},\mathcal{X}'} \rangle$ , we then say that the elements of the directed limit:

$$\varinjlim_{\mathcal{X}} \operatorname{VCar}(\mathcal{X}),$$

are called vertical Cartier b-divisors.

For each test configuration  $\mathcal{X}$  we can define a natural map:

$$\operatorname{PL}_+(X^{\beth}) \to \operatorname{VCar}(\mathcal{X}),$$

that assigns to each  $\varphi \in PL_+$  the vertical divisor given by:

$$\sum_{\substack{\text{irred}\\E \subseteq \mathcal{X}_0}} b_E \varphi(v_E) E.$$
(2.6.1)

We show now that these maps glue well to define an universal one to the direct limit  $\varinjlim_{\mathcal{X}} \operatorname{VCar}(\mathcal{X})$ .

**Lemma 2.6.3.** The collection of the above mentioned maps induces the mapping:

$$\operatorname{PL}^+(X^{\beth}) \to \varinjlim_{\mathcal{X}} \operatorname{VCar}(\mathcal{X}).$$

*Proof.* Let  $\varphi \in PL^+(X^{\beth})$ , we show that for a cofinal set of test configurations, and  $\mu \colon \mathcal{X}' \to \mathcal{X}$  a morphism of such test configurations:

$$\mu^* \left( \sum_{\substack{\text{irred} \\ E \subseteq \mathcal{X}_0}} b_E \, \varphi(v_E) E \right) = \sum_{\substack{E' \subseteq \mathcal{X}_0'}} b_{E'} \, \varphi(v_{E'}) E'.$$

After scaling, we may assume that  $\varphi = \varphi_{\mathfrak{a}}$  for some flag ideal  $\mathfrak{a}$ . The set of test configurations that dominate the normalized blow-up of  $X \times \mathbb{P}^1$  along  $\mathfrak{a}$  is cofinal, and thus we will suppose that  $\mathcal{X}$  is in this set. Let G be the effective divisor induced by  $\mathfrak{a}$  on  $\mathcal{X}$ :

$$\mathcal{O}_{\mathcal{X}}(-G) = \mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}}.$$
(2.6.2)

Therefore,

$$\sum_{E \subseteq \mathcal{X}_0} b_E \varphi(v_E) E = -G.$$

Writing  $\mathcal{X}'_0 = \sum_{E' \subseteq \mathcal{X}'_0} b_{E'} E'$  as the irreducible decomposition, it follows that:

$$\mu^* \left( \sum_{E \subseteq \mathcal{X}_0} b_E \varphi(v_E) E \right) = \mu^* (-G) = \sum_{E' \subseteq \mathcal{X}'_0} \operatorname{ord}_{E'} (\mathcal{O}(\mu^*G)) E'$$
$$= \sum_{E' \subseteq \mathcal{X}'_0} -\operatorname{ord}_{E'} (\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'}) E'$$
$$= \sum_{E' \subseteq \mathcal{X}'_0} -b_{E'} v_{E'} (\mathfrak{a}) E'$$
$$= \sum_{E' \subseteq \mathcal{X}'_0} b_{E'} \varphi(v_{E'}) E',$$

concluding the proof.

Theorem 2.6.4. The above map induces an isomorphism

$$\operatorname{PL}(X^{\beth}) \simeq \varinjlim_{\mathcal{X}} \operatorname{VCar}(\mathcal{X}).$$

*Proof.* Since the map is additive, we have a unique linear extension:

$$\operatorname{PL}(X^{\beth}) \to \varinjlim_{\mathcal{X}} \operatorname{VCar}(\mathcal{X}).$$

We now construct its inverse.

Let  $\mathcal{X}$  be a test configuration of X. By Remark 2.5.2 we can suppose that there exists a morphism of test configurations  $\mu \colon \mathcal{X} \to X \times \mathbb{P}^1$  that is projective, that is we can embbed  $\mathcal{X}$ :



making the diagram commute, where  $p_i$  is the *i*-th coordinate projection. Then the set:

$$\operatorname{AVCar}(\mathcal{X}) \doteq \{ D \in \operatorname{VCar}(\mathcal{X}) | -D \text{ is } \mu \text{-very ample} \}$$

is non-empty, since  $p_2^*(mL)|_{\mathcal{X}} \in \operatorname{AVCar}(\mathcal{X})$ , for L an ample line bundle on  $\mathbb{P}^{\ell}$  and for  $m \gg 0$ . Moreover,  $\operatorname{AVCar}(\mathcal{X})$  is a semigroup that Q-spans  $\operatorname{VCar}(\mathcal{X})$ .

For each  $D \in \text{AVCar}(\mathcal{X})$ , -D is  $\mu$ -globally generated, which means that there exists  $\mathfrak{b}$ , a fractional ideal sheaf of  $\mathcal{O}_{X \times \mathbb{P}^1}$ , such that

$$\mathcal{O}_{\mathcal{X}}(-D) = \mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}.$$
(2.6.3)

Since D is a vertical divisor implies that we can suppose  $\mathfrak{b} \in \mathcal{F}$ , and hence we define  $\varphi_D \doteq -\varphi_{\mathfrak{b}}$ . To see that  $\varphi_D$  is well defined, is enough to observe that if

$$\mathcal{O}_{\mathcal{X}}(-D) = \mathfrak{b}' \cdot \mathcal{O}_{\mathcal{X}} \tag{2.6.4}$$

then by Corollary 2.2.3  $\overline{\mathfrak{b}} = \overline{\mathfrak{b}'}$ , which implies

$$\varphi_{\mathfrak{b}} = \varphi_{\overline{\mathfrak{b}}} = \varphi_{\overline{\mathfrak{b}'}} = \varphi_{\mathfrak{b}}$$

by Lemma  $2.2.4^8$ .

27

<sup>&</sup>lt;sup>8</sup>The proof of Lemma 2.2.4 applies, since for every flag ideal  $\mathfrak{a}$  the function  $\log|\mathfrak{a}|$  is finite valued on  $(X \times \mathbb{P}^1)^{\neg}_{\mathbb{C}^*}$ .

Let us now check that  $\varphi \colon \operatorname{AVCar}(\mathcal{X}) \to \operatorname{PL}(X^{\beth})$  is additive. Pick  $D, D' \in \operatorname{AVCar}(\mathcal{X})$ and write

$$\mathcal{O}_{\mathcal{X}}(-D) = \mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}} \quad \& \quad \mathcal{O}_{\mathcal{X}}(-D') = \mathfrak{c} \cdot \mathcal{O}_{\mathcal{X}},$$

which implies

$$\mathcal{O}_{\mathcal{X}}(-D-D') = \mathcal{O}_{\mathcal{X}}(-D) \cdot \mathcal{O}_{\mathcal{X}}(-D') = (\mathfrak{b} \cdot \mathfrak{c}) \cdot \mathcal{O}_{\mathcal{X}}$$

and thus

$$\varphi_{D+D'} = -\varphi_{\mathfrak{b}\cdot\mathfrak{c}} = -\varphi_{\mathfrak{b}} - \varphi_{\mathfrak{c}} = \varphi_D + \varphi_{D'}$$

Again we can extend  $\varphi$  uniquely to VCar( $\mathcal{X}$ ) by linearity.

Observe that this definition does not depend on  $\mathcal{X}$ , in the sense that if  $\mu \colon \mathcal{X}' \to \mathcal{X}$  is a morphism of test configurations, and  $D' \doteq \mu^* D \subseteq \mathcal{X}'$ , then we have

$$\mathcal{O}_{\mathcal{X}'}(-D') = (\mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}) \cdot \mathcal{O}_{\mathcal{X}'} = \mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}'}.$$
(2.6.5)

Hence  $\varphi_{D'} = \varphi_D$ .

This defines  $^{9}$  a linear map

$$\varinjlim_{\mathcal{X}} \operatorname{VCar}(\mathcal{X}) \to \operatorname{PL}(X^{\beth})$$

which is the inverse of (2.6.1).

We thus can conclude the proof of Theorem 2.3.6.

Proof of Theorem 2.3.6. If  $\varphi \in PL(X^{\beth})$  is such that

$$\varphi(v) = 0, \quad \text{for every } v \in X^{\text{div}}.$$

Then, using Theorem 2.5.5 we have that for all test configurations  $\mathcal{X}'$  and all prime vertical divisors  $E' \in \mathrm{VCar}(\mathcal{X}')$ 

$$\varphi(v_{E'}) = 0$$

in particular if  $D \in VCar(\mathcal{X})$  is a vertical divisor, such that  $\varphi_D = \varphi$ , writing

$$D = \sum_{\substack{E \subseteq \mathcal{X}_0}} \varphi_D(v_E) E,$$

we will have D = 0, and thus  $\varphi = 0$ .

### 3. DUAL COMPLEXES AND LOG DISCREPANCY

From this point on X will be a compact complex manifold.

3.1. Non-archimedean as a limit of tropical. In the algebraic setting it is known that the Berkovich analytification corresponds to taking a limit of Tropical complexes, known as the *Dual Complex*, associated to a test configurations. See [BJ22, Appendix A] for a version in the trivially valued case. In this section we will show the analogous result in our transcendental setting.

28

<sup>&</sup>lt;sup>9</sup>The set of test configurations obtained by a sequence of blow-ups is cofinal.

3.1.1. Contruction of the dual complex. Let  $\mathcal{X}$  a smooth snc test configuration for X. Let  $\mathcal{X}_0 = \sum_i b_i E_i$  be the decomposition of the central fiber in it's irreducible components.

Then  $(\mathcal{X}, \mathcal{X}_{0, red})$  is a snc reduced birational model of  $\mathcal{X}_{triv}$ . Recall from Section 2.3.1 that we can then construct  $\hat{\Delta}_{\mathcal{X}} \doteq \hat{\Delta}(\mathcal{X}, \mathcal{X}_{0, red})$ .

Now, we will construct a simplicial complex,  $\Delta_{\mathcal{X}}$ , as a sort of compact representative of  $\hat{\Delta}_{\mathcal{X}}$ . For each "cone" face,  $\hat{\sigma}_Z \cong (\mathbb{R}_+)^J$ , of  $\hat{\Delta}_{\mathcal{X}}$  we will associate a "simplex" face,  $\sigma_Z$ , of  $\Delta_{\mathcal{X}}$  given by the equation  $\sum b_i w_i = 1$ , that is:

$$\sigma_Z \doteq \left\{ w \in \hat{\sigma}_Z \cong (\mathbb{R}_+)^J \mid \sum_{i \in J} b_i w_i = 1 \right\}.$$

Given a test configuration  $\mathcal{X}$ , we have a natural map:

$$p_{\mathcal{X}} \colon X^{\beth} \to \Delta_{\mathcal{X}} \tag{3.1.1}$$

defined by  $p_{\mathcal{X}}(v) = (v(E_i)) \in \mathbb{R}^J_+$ , where the latter that corresponds to the stratum Z, the smallest one that contains  $Z(v, \mathcal{X})$  the central variety of v on  $\mathcal{X}$ .

3.1.2. Morphisms. Let  $\mathcal{X}, \mathcal{X}'$  be test configurations of  $X, \mu \colon \mathcal{X} \to \mathcal{X}'$  a test configuration morphism between them, and  $\sum b_i E_i, \sum c_j E'_j$  be the decomposition in irreducible components of  $\mathcal{X}_0, \mathcal{X}'_0$  respectively, then clearly:

$$\operatorname{Supp}(\mathcal{X}_0) \subseteq \operatorname{Supp}(\mu^* \mathcal{X}'_0).$$

In particular, we can write  $\mu^* E'_j = \sum_i d^i_j E_i$ , for  $D_j = (d^1_j, \ldots, d^M_j) \in \mathbb{R}^M$ , and we define the map:

$$r_{\mathcal{X},\mathcal{X}'} \colon \Delta_{\mathcal{X}} \longrightarrow \Delta_{\mathcal{X}'}$$
$$(\mathbb{R}_+)^J \cong \sigma_Z \ni w \mapsto r_{\mathcal{X},\mathcal{X}'}(w) \in \sigma_{Z'} \cong (\mathbb{R}_+)^{J'_w},$$

for  $J'_w \doteq \{j \in I \mid d^i_j \neq 0 \text{ for some } i \in J\}$ , given by:

$$r_{\mathcal{X},\mathcal{X}'}(w) \doteq \left(\sum d_j^i w_i\right)_{j \in J'_w}.$$

Since the snc test configurations form a directed poset we can take the projective limit,

$$\Delta \doteq \lim_{\substack{\mathcal{X} \text{ snc}}} \Delta_{\mathcal{X}},$$

and the family of maps  $(p_{\mathcal{X}})_{\mathcal{X}}$  induces an injective<sup>10</sup> continuous map:

$$p\colon X^{\beth} \to \Delta.$$

**Theorem 3.1.1.** The map  $p: X^{\square} \to \Delta$  is a homeomorphism.

To get this, we'll see that, just like  $X^{\beth}$ ,  $\Delta$  has a PL structure, that will be isomorphic to the PL structure on  $X^{\beth}$ .

3.1.3. *PL Functions.* There is a natural class of functions defined on  $\Delta$ , the ind-type set of *piecewise linear functions.* That is, the set of real valued functions that, on a complex  $\Delta_{\mathcal{X}}$ , are Q-piecewise linear:

$$\operatorname{PL}(\Delta) \doteq \bigcup_{\mathcal{X}} (\pi_{\mathcal{X}})^* \operatorname{PL}(\Delta_{\mathcal{X}}),$$

for  $\pi_{\mathcal{X}} \colon \Delta \to \Delta_{\mathcal{X}}$  the canonical projection.

<sup>&</sup>lt;sup>10</sup>this is equivalent to  $PL(X^{\supset})$  separating points on  $X^{\supset}$ 

After going to a higher model, we can assume that the functions are rationally affine on each face of the associated dual complex  $\Delta_{\mathcal{X}}$ , and hence we have:

$$\operatorname{PL}(\Delta) = \varinjlim_{\mathcal{X}} \operatorname{Aff}_{\mathbb{Q}}(\Delta_{\mathcal{X}}).$$

Now, observe that  $\operatorname{VCar}(\mathcal{X})_{\mathbb{Q}} \cong \operatorname{Aff}_{\mathbb{Q}} \Delta_{\mathcal{X}}$ , where the isomorphism is given by:

$$\operatorname{Aff}_{\mathbb{Q}}\Delta_{\mathcal{X}} \ni f \mapsto \sum b_i f(e_i) E_i.$$
(3.1.2)

Taking the limit we get:

$$\operatorname{PL}(\Delta) = \lim_{\mathcal{X}} \operatorname{Aff}_{\mathbb{Q}} \Delta_{\mathcal{X}} \cong \lim_{\mathcal{X}} \operatorname{VCar}(\mathcal{X}) \cong \operatorname{PL}(X^{\beth}).$$
(3.1.3)

Lemma 3.1.2. The map

$$p\colon X^{\beth} \to \Delta$$

is an ismorphism of PL structures<sup>11</sup>.

*Proof.* We need to check that if  $\eta: \operatorname{PL}(\Delta) \to \operatorname{PL}(X^{\beth})$  is the ismorphism of Equation (3.1.3), then, for  $f \in \operatorname{PL}(\Delta)$  and  $v \in X^{\square}$ :

$$\eta(f)(v) = f(p(v)).$$
(3.1.4)

By Theorem 2.3.6 it is enough to check (3.1.4) for  $v \in X^{\text{div}}$ .

Now, given  $f \in PL(\Delta)$ , and  $v \in X^{div}$ , let  $\mathcal{X}$  be a smooth test configuration such that:

- the function  $f|_{\Delta_{\mathcal{X}}}$  is rationally affine, for  $\Delta_{\mathcal{X}}$  the associated dual complex;
- decomposing the central fiber  $\mathcal{X}_0 \doteq \sum_i b_i E_i$ , we have  $v = v_{E_1}$ .

Then

$$f(p(v)) = f(v(E_1), v(E_2), \dots, v(E_k)) = v(E_1)f(e_1) = f(e_1) = \eta(f)(v_{E_1})$$

where the last equality is given by (3.1.2) together with (2.6.1).

Now, we will prove that the isomorphism (3.1.3) induces a homeomorphism

$$X^{\beth} \stackrel{p}{\cong} \Delta.$$

To do that, as mentioned before, we will use an analogue of the Gelfand transform, to show that  $X^{\beth}$  and  $\Delta$  can be seen as the "tropical spectra" of  $PL(X^{\beth})$  and  $PL(\Delta)$  respectively. Then, since the map

$$p\colon X^{\beth} \to \Delta$$

is an isomorphism of PL structures, p will be a homeomorphism.

3.1.4. *Tropical Gelfand transform*. Let's recall some definitions from Section 1.2, and from Appendix A.

Let  $\mathcal{A}$  be a tropical algebra, that is a vector space together with a semi-ring operation, which we will denote by  $\{\cdot, \cdot\}$ , that makes  $(\mathcal{A}, \{\cdot, \cdot\}, +)$  a semi-ring. The *tropical spectrum* of  $\mathcal{A}$  is the topological space given by<sup>12</sup>

TropSpec 
$$\mathcal{A} = \{\varphi \in \mathcal{A}^* | \varphi(\{f, g\}) = \max\{\varphi(f), \varphi(g)\}\},$$
 (3.1.5)

where  $\mathcal{A}^*$  denotes the algebraic dual. We endow TropSpec  $\mathcal{A}$  with pointwise convergence topology.

30

<sup>&</sup>lt;sup>11</sup>See Appendix A.

 $<sup>^{12}</sup>$ cf. Lemma A.2.6

**Proposition 3.1.3.** Let K be a compact Hausdorff topological space, and  $\mathcal{A} \subseteq C^0(K, \mathbb{R})$ a dense linear subspace, containing all the constants, that is stable by max. Then  $\mathcal{A}$  is subtroipcal algebra of  $C^0(K, \mathbb{R})$ , and the map:

$$\delta \colon K \to (\operatorname{TropSpec} \mathcal{A}), \quad \delta_x(f) = f(x),$$

induces the homeomorphism:

$$[\delta] \colon K \to (\operatorname{TropSpec}(\mathcal{A}) \setminus \{0\}) / \mathbb{R}_{>0}.$$

*Proof.* It is clear that  $\mathcal{A}$  is tropical algebra with the max (or min as it also preserves minima) and sum as operations.

Moreover, the map  $\delta$  is clearly continous and injective, therefore it suffices to prove that  $[\delta]$  is surjective, since K is compact.

Let  $\varphi \in \operatorname{TropSpec} \mathcal{A} \setminus \{0\}$ , then

$$\varphi(|f|) = \varphi(\max\{f, -f\}) = \max\{\varphi(f), -\varphi(f)\} = |\varphi(f)|$$
(3.1.6)

hence  $\varphi(1) > 0$ , thus we can suppose that  $\varphi(1) = 1$ , and

$$\begin{split} |\varphi(f)| &= \varphi(|f|) \leq \max\{\varphi(|f|), \varphi(\|f\|_{\infty})\} \\ &= \varphi\left(\max\{|f|, \|f\|_{\infty}\}\right) = \varphi(\|f\|_{\infty}) = \|f\|_{\infty} \cdot 1 \end{split}$$

Hence  $\varphi$  can be extended to a continuous linear functional on  $C^0(K)$ . That is,  $\varphi$  is a signed measure on K. By (3.1.6),  $\varphi$  is actually a positive measure.

Let  $x \in \operatorname{supp} \varphi$ , we will show that  $\varphi = \delta_x$ . To do that we will just prove that ker  $\delta_x = \ker \varphi$ , and the equality will follow since  $\varphi(1) = 1 = \delta_x(1)$ .

If  $f \in \ker \varphi$ , then, by Equation (3.1.6),  $|f| \in \ker \varphi$ . Since  $\varphi$  is a positive measure, and  $|f| \ge 0$ , we get  $f = 0 \varphi$ -almost everywhere. Therefore, since f is continuous, the restriction:

$$f|_{\operatorname{supp}\varphi} = 0$$

In particular, f(x) = 0, and thus  $f \in \ker \delta_x$ . Since  $\operatorname{codim} \ker \varphi = \operatorname{codim} \ker \delta_x$ , we conclude.

Proof of Theorem 3.1.1. The map

$$p\colon X^{\beth} \to \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}}$$

induces the isomorphism:

$$\eta \colon \varinjlim_{\mathcal{X}} \operatorname{Aff}_{\mathbb{Q}} \Delta_{\mathcal{X}} \to \operatorname{PL}(X^{\beth}),$$

and therefore we get a homeomorphism:

$$\eta^*$$
: TropSpec  $\left( \varinjlim_{\mathcal{X}} \operatorname{Aff}_{\mathbb{Q}} \Delta_{\mathcal{X}} \right) \to \operatorname{TropSpec} \left( \operatorname{PL}(X^{\beth}) \right),$ 

given by  $\eta^*(\delta_x)(f) = \delta_x(\eta(f)) = \eta(f)(x) = f(p(x)) = \delta_{p(x)}(f)$  which means that the map

$$X^{\beth} \xrightarrow{\delta} \operatorname{TropSpec}\left(\operatorname{PL}(X^{\beth})\right) \xrightarrow{\eta^{*}} \operatorname{TropSpec}\left(\varinjlim_{\mathcal{X}} \operatorname{Aff}_{\mathbb{Q}} \Delta_{\mathcal{X}}\right) \xrightarrow{\delta^{-1}} \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}}$$
(3.1.7)

is given by  $x \mapsto p(x)$ . Hence, p is a homeomorphism.

# 3.2. Log discrepancy on $X^{\beth}$ .

Log discrepancy over X. Let  $F \subseteq Y \xrightarrow{\pi} X$  be an irreducible divisor, and  $v = r \operatorname{ord}_F$  for some  $r \in \mathbb{Q}_+$  We define the log discrepancy of v to be the quantity

$$A_X(v) \doteq r \cdot \left(1 + \operatorname{ord}_F(K_{Y/X})\right). \tag{3.2.1}$$

This gives us a function  $A_X \colon X^{\text{div}} \to \mathbb{Q}$ , which will be called the *log discrepancy over* X. If r = 1 we sometimes denote  $A_X(F) \doteq A_X(\text{ord}_F)$ .

By some standard calculations, as in [Kol97, Section 3], the restriction of  $A_X$  to (the rational points of) each face of  $\hat{\Delta}(Y, B) \subset X^{\beth}$  is linear, and hence we can extend it to  $\hat{\Delta}(Y, B)$  by linearity.

Again by [Kol97, Section 3] if  $(Y', B') \xrightarrow{\mu} (Y, B)$  is a snc reduced birational projective morphism over (Y, B), then the piecewise linear function induced by the pullback:

$$r_{\mu} \colon \hat{\Delta}(Y, B) \to \hat{\Delta}(Y', B')$$

satisfies the inequality

$$A_X \circ r_\mu \le A_X. \tag{3.2.2}$$

Log discrepancy over  $X \times \mathbb{P}^1$ . The same log discrepancy defined on the previous section makes sense for  $X \times \mathbb{P}^1$ . We study now the relationship between  $A_X$ , and  $A_{X \times \mathbb{P}^1} \circ \sigma$ , where  $\sigma$  is the Gauss extension.

Let  $F \subseteq Y \xrightarrow{\pi} X$  be a prime divisor, and  $\operatorname{ord}_F$  the associated divisorial valuation. Then, consider the divisors:

$$\begin{array}{cccc} F \times \mathbb{P}^1 \subseteq Y \times \mathbb{P}^1 & \& & Y \times \{0\} \subseteq Y \times \mathbb{P}^1 \\ & & & \downarrow \\ & & & & \downarrow \\ & & & X \times \mathbb{P}^1 & & & X \times \mathbb{P}^1. \end{array}$$

A direct calculation gives us that  $\sigma(\operatorname{ord}_F)$  is monomial with respect to  $F \times \mathbb{P}^1$  and  $Y \times \{0\}$ , with associated weights (1, 1). Therefore, using linearity of  $A_{X \times \mathbb{P}^1}$ , we get:

$$A_{X \times \mathbb{P}^1} \left( \sigma(\operatorname{ord}_F) \right) = A_{X \times \mathbb{P}^1} (F \times \mathbb{P}^1) + A_{X \times \mathbb{P}^1} (Y \times \{0\})$$
  
=  $A_X(F) + 1 = A_X(\operatorname{ord}_F) + 1.$  (3.2.3)

Let  $\mathcal{X}$  be a smooth test configuration. Like we did in Section 3.1.1, one can associate  $\mathbb{C}^*$ -equivariant monomial valuations on  $(X \times \mathbb{P}^1)^{\neg}_{\mathbb{C}^*}$  to points  $w \in \sigma_Z \subseteq \Delta_{\mathcal{X}}$ . Again, the rational points on  $\Delta_{\mathcal{X}}$  correspond to points in  $(X \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*}$ . By the same the reasoning as before we get:

(1) Let  $w \in (\sigma_Z)_{\mathbb{Q}} \subseteq (\Delta_{\mathcal{X}})_{\mathbb{Q}}$ , and val(w) the associated valuation satisfies:

$$A_{X \times \mathbb{P}^1}(\operatorname{val}(w)) = \sum w_i A_{X \times \mathbb{P}^1}(E_i).$$

(2) For  $\mathcal{X}' \xrightarrow{\mu} \mathcal{X}$  a morphism of test configurations, we have

$$A_{X \times \mathbb{P}^1} \circ p_{\mathcal{X}} \le A_{X \times \mathbb{P}^1}$$

on  $(\Delta_{\mathcal{X}'})_{\mathbb{Q}}$ .

Therefore, by (1) we can extend by linearity  $A_{X \times \mathbb{P}^1}$  to  $\Delta_{\mathcal{X}}$ , and, by (2), define the limit:

$$A_{X \times \mathbb{P}^1} \colon X^{\beth} \to \mathbb{R} \cup \{+\infty\}$$
(3.2.4)

as the sup  $A_{X \times \mathbb{P}^1}(v) \doteq \sup_{\mathcal{X}} A(p_{\mathcal{X}}(v)).$ 

The function on (3.2.4) will be called the *log discrepancy*, and from here on will be denoted by  $A: X^{\square} \to \mathbb{R} \cup \{+\infty\}$ .

**Remark 3.2.1.** It is clear to see that from the definition of the log discrepancy, if  $\mathcal{X}$  is a test configuration, and  $p_{\mathcal{X}}$  is the function defined in Section 3.1, then  $A \circ p_{\mathcal{X}}$  is a PL function.

### 4. Non-Archimedean plurisubharmonic functions

From now on, X will be a compact Kähler manifold, with a fixed Kähler class  $\alpha \in Pos(X)$ .

4.1. Plurisubharmonic PL functions. Let's denote by  $\mathcal{X}_{triv} \doteq X \times \mathbb{P}^1$  the trivial configuration, and by  $p_1: \mathcal{X}_{triv} \to X$  the first projection. Given  $\beta \in H^{1,1}(X)$ , we then denote

$$\beta_{\mathcal{X}_{\text{triv}}} \doteq p_1^* \beta \in H^{1,1}(X \times \mathbb{P}^1)$$

More generally, given any test configuration that  $\mu$ -dominates  $X \times \mathbb{P}^1$ , we denote

$$\beta_{\mathcal{X}} \doteq \mu^* \beta_{\mathcal{X}_{\mathrm{triv}}}.$$

**Remark 4.1.1.** In [SD18, DR17b] the authors introduce, independently, the notion of cohomological test configurations, which are generalizations –to the transcendental setting– of the usual algebraic test configurations for a polarized manifold (X, L).

For them, a cohomological test configuration is a test configuration  $\mathcal{X}$  together with a  $\mathbb{C}^*$ -invariant Bott-Chern cohomology class  $\mathcal{A} \in H^{1,1}_{\mathrm{BC}}(\mathcal{X})$  such that away from the central fiber:

$$\mathcal{A}|_{\mathcal{X}^*} = h^* \alpha_{\mathcal{X}}$$

for  $h: \mathcal{X}^* \to X \times (\mathbb{P}^1 \setminus \{0\})$  the  $\mathbb{C}^*$ -equivariant biholomorphism.

By [SD18, Proposition 3.10], the data of a cohomological test configuration is the same of a test configuration together with the choice of a vertical divisor D, i.e.

$$\mathcal{A} = \alpha_{\mathcal{X}} + D \tag{4.1.1}$$

for  $D \in \operatorname{VCar} \mathcal{X}$ .

**Definition 4.1.2.** Let  $\varphi \in PL(X^{\beth})$  we say that  $\varphi$  is  $\alpha$ -plurisubharmonic if given a dominating test configuration  $\mathcal{X}$  with  $D \in VCar(\mathcal{X})$  such that  $\varphi = \varphi_D$ , we have

$$\alpha_{\mathcal{X}} + D$$
 is nef relatively to  $\mathbb{P}^1$ . (4.1.2)

Since the pullback by an holomorphic map of a (1,1)-class is nef if and only if the (1,1)-class itself is nef, this definition does not depend on  $\mathcal{X}$ .

We will denote the set of  $\alpha$ -psh functions by  $PL \cap PSH(\alpha)$ . Moreover, we denote by  $\mathcal{H}(\alpha)$  the set of functions  $\varphi \in PL(X^{\neg})$  such that there exists a dominating test configuration  $\mathcal{X}$  and a vertical divisor  $D \in VCar(\mathcal{X})$  satisfying:

 $\varphi = \varphi_D, \quad \alpha_{\mathcal{X}} + D \text{ is Kähler relatively to } \mathbb{P}^1.$ 

**Remark 4.1.3.** In the standard algebraic setting, the set of non-archimedean Fubini– Study metrics is usually denoted by  $\mathcal{H}^{NA}$ . Here  $\mathcal{H}(\alpha)$  will play the role of this set in our more general context. Note, however, that for an algebraic variety,  $\mathcal{H}(\alpha)$  it is not the set Fubini–Study functions, the latter is only a subset:  $\mathcal{H}^{NA} \subseteq \mathcal{H}(\alpha)$ .

**Proposition 4.1.4.** Let  $\varphi, \psi \in PL(X^{\beth}) \cap PSH(\alpha)$ , and  $f: X \to Y$  be a finite holomorphic map, then the following properties hold:

- (1)  $f^*\varphi \in \mathrm{PL} \cap \mathrm{PSH}(f^*\alpha);$
- (2)  $\varphi + c$ , and  $t \cdot \varphi$  lie in  $PL \cap PSH(\alpha)$  for  $c \in \mathbb{R}$  and  $t \in \mathbb{Q}$ ;
- (3)  $\max\{\varphi, \psi\} \in PL \cap PSH(\alpha)$ .

While proof of items (1) and (2) is essentially the same as in the algebraic trivially valued case, cf. [BJ22, Proposition 3.6], the proof of item (3) is different and relies on the analysis of singularities of psh functions of Lemma 5.0.2.

*Proof.* Let  $\mathcal{X}$  be a test configuration  $\mu$ -dominating the trivial one, such that

$$\varphi = \varphi_D \quad \& \quad \psi = \varphi_E$$

for some  $D, E \in \operatorname{VCar}(\mathcal{X})$ .

For (1) it is enough to observe that the pull-back of a PL function is PL and that pull-back of a nef class is nef.

For the first part of (2) it is enough to observe that:

- $\varphi + c = \varphi_{D+c\mathcal{X}_0};$
- On the quotient

$$H^{1,1}(\mathcal{X})/\mu^*(H^{1,1}(\mathbb{P}^1))$$

we have  $[\alpha + D] = [\alpha + D + c\mathcal{X}_0]$ , and hence

$$D + \alpha + c\mathcal{X}_0 \in \operatorname{Nef}(\mathcal{X}/\mathbb{P}^1) \iff D + \alpha \in \operatorname{Nef}(\mathcal{X}/\mathbb{P}^1).$$

If  $t \in \mathbb{N}$ , then it is enough to observe that, like in the trivially valued case [BJ22, Proposition 3.6],

$$\varphi_{D_t} = t \cdot \varphi_D$$

where  $D_t = \mu_t^* D$  is given by the base change

The result follows from (1), the general case  $t \in \mathbb{Q}$  follows from this one, since a cohomology class is nef iff its pullback by a finite branched covering is.

For item (3), let's suppose  $\varphi, \psi \in \mathcal{H}(\alpha)$ , the general case will follow from an approximation argument, cf. Theorem 4.1.5 below.

Now, let c > 0 be large enough so that  $D' \doteq -D + c\mathcal{X}_0$  and  $E' \doteq -E + c\mathcal{X}_0$  are effective, we have:

$$\max\{\varphi_{-D'}, \varphi_{-E'}\} = \max\{\varphi, \psi\} - c,$$

and thus it is  $\alpha$ -psh iff max{ $\varphi, \psi$ } is  $\alpha$ -psh.

Moreover, let  $\mathcal{X}'$  be a test configuration  $\nu$ -dominating  $\mathcal{X}$ , such that

$$\varphi_G = \max\{\varphi_{-D'}, \varphi_{-E'}\}$$

for  $G \in \text{VCar}(\mathcal{X}')$ , then we observe that  $\mathcal{O}_{\mathcal{X}'}(-G) = \nu^*[\mathcal{O}_{\mathcal{X}}(D') + \mathcal{O}_{\mathcal{X}}(E')]$ , and the result follows from Lemma 5.0.2.

We will show that the set of  $\alpha$ -psh functions is stable under decreasing limits.

**Theorem 4.1.5.** Let  $\varphi \in PL(X^{\beth})$ , and  $(\varphi_{\lambda})_{\lambda \in \Lambda}$  be a net of PL  $\alpha$ -psh functions such that  $\varphi_{\lambda}(v) \to \varphi(v), \ \forall v \in X^{\operatorname{div}}$ 

then  $\varphi$  is PL  $\alpha$ -psh function.

*Proof.* Let  $\mathcal{X}_{\lambda}$ , and  $\mathcal{X}$  be snc test configurations together with morphisms of test configurations  $\mu_{\lambda} \colon \mathcal{X}_{\lambda} \to \mathcal{X}$ , and  $\nu \colon \mathcal{X} \to X \times \mathbb{P}^1$ , such that there exist vertical divisors  $D \in \mathrm{VCar}(\mathcal{X})$ , and  $D_{\lambda} \in \mathrm{VCar}(\mathcal{X}_{\lambda})$  satisfying:

$$\varphi = \varphi_D$$
, and  $\varphi_\lambda = \varphi_{D_\lambda}$ .

To prove that  $D + \alpha_{\mathcal{X}}$  is nef relatively to  $\mathbb{P}^1$  it is enough to show that for every irreducible -hence smooth- component of the central fiber,  $E \subseteq \mathcal{X}_0$ , the restriction  $(D + \alpha_{\mathcal{X}})|_E$  is in Nef(E). This follows from the simple observation that for  $\tau \neq 0$  we have:

$$D_{|_{\mathcal{X}_{\tau}}} = 0 \quad \& \quad (\alpha_{\mathcal{X}})_{|_{\mathcal{X}_{\tau}}} = h_{\tau}^* \alpha \in \operatorname{Nef}(\mathcal{X}_{\tau}),$$

where  $h_t: \mathcal{X}_{\tau} \to X$  is the biholomorphism provided by the  $\mathbb{C}^*$ -action.

Let then  $Y^d \subseteq E$  be a *d*-dimensional subvariety of E, and  $\gamma \in \text{Pos}(\mathcal{X})$  a Kähler class, by Demailly-Paun numeric characterization of nefness, it suffices to show that:

$$\int_{Y} (D + \alpha_{\mathcal{X}}) \wedge \gamma^{d-1} \ge 0.$$
(4.1.4)

To simplify notation, we rewrite the left hand side:

$$\int_{Y} (D + \alpha_{\mathcal{X}}) \wedge \gamma^{d-1} = [Y] \cdot (D + \alpha_{\mathcal{X}}) \cdot \gamma^{d-1}.$$
(4.1.5)

We can suppose that Y is invariant by the C<sup>\*</sup>-action. Indeed, since  $E \subseteq \mathcal{X}_0$  is irreducible, it is itself invariant. Therefore, denoting  $Y_{\tau} \doteq \tau \cdot Y$ , we get by compacity of each component of the space of effective cycles on E, cf. [HS74, Fuj78], that the limit  $\lim_{\tau\to 0} Y_{\tau}$  exists as an effective cycle  $\sum a_Z Z$ , where the components Z are C<sup>\*</sup>-invariant. Moreover, since C<sup>\*</sup> acts trivially on cohomology, we observe that  $[Y_{\tau}] = [Y]$ , and:

$$[Y] \cdot (D + \alpha_{\mathcal{X}}) \cdot \gamma^{d-1} = [Y_{\tau}] \cdot \rho(\tau)^* (D + \alpha_{\mathcal{X}}) \cdot \rho(\tau)^* \gamma^{d-1} \to \sum a_Z[Z] \cdot (D + \alpha_{\mathcal{X}}) \cdot \gamma^{d-1},$$

for  $\rho$  the  $\mathbb{C}^*$ -action on  $\mathcal{X}$ . Replacing Y for Z we get the  $\mathbb{C}^*$ -invariance.

Let  $b: \mathcal{X}' \to \mathcal{X}$  be the blow-up of  $\mathcal{X}$  along Y. Since Y is  $\mathbb{C}^*$ -invariant  $\mathcal{X}'$  is a test configuration. Let  $F \subseteq \mathcal{X}'$  be the exceptional divisor, and consider the positive current, of bi-dimension (d, d), given by:

$$T \doteq \delta_F \wedge \omega^{n-d},$$

where  $\delta_F$  denotes the current of integrantion on F, and  $\omega \in \mathcal{K}(\mathcal{X}')$  a Kähler form.

Consider now  $b_*(T)$ , by definition this is again a positive current of bi-dimension (d, d), with support supp  $b_*(T) \subseteq b(\operatorname{supp}(T)) = b(F) = Y$ . By Demailly's support theorem every current of bi-dimension (d, d) supported on an irreducible cycle of dimension d must be a multiple of the current of integration over that cycle, which implies that the cohomology classes  $[b_*(T)] = a[Y]$ , for  $a \geq 0$ . Choosing  $\eta$  to be a Kähler form on  $\mathcal{X}$  such that  $\omega - b^*\eta$ is positive on the fibers of b, we have:

$$b_*T \cdot \eta = T \cdot b^*\eta \ge \int_F (\omega - b^*\eta)^{n-d} \wedge (b^*\eta)^d > 0,$$

thus  $[b_*T] \neq 0 \implies a > 0$ .

Hence, Equation (4.1.5) becomes:

$$\frac{1}{a}[b_*(T)] \cdot (D + \alpha_{\mathcal{X}}) \cdot \gamma^{d-1} = \frac{1}{a}[T] \cdot b^*(D + \alpha_{\mathcal{X}}) \cdot b^* \gamma^{d-1}$$
$$= \frac{1}{a}[F] \cdot \alpha^{n-d} \cdot (D_{\mathcal{X}'} + \alpha_{\mathcal{X}'}) \cdot \gamma^{d-1}_{\mathcal{X}'}$$

where the second equality holds by the projection formula, and  $D_{\mathcal{X}'} \doteq b^* D$ .

Now, let  $\mathcal{X}'_{\lambda}$  be a test configuration that dominates both  $\mathcal{X}_{\lambda}$  and  $\mathcal{X}'$ 

$$\begin{array}{ccc} \mathcal{X}_{\lambda} \xleftarrow{b_{\lambda}} \mathcal{X}'_{\lambda} \\ \downarrow^{\mu_{\lambda}} & \downarrow^{\nu_{\lambda}} \\ \mathcal{X} \xleftarrow{b} \mathcal{X}'. \end{array}$$

Denoting  $F_{\lambda} \doteq \nu_{\lambda}^* F$ , and  $\omega_{\lambda} \doteq \nu_{\lambda}^* \omega$ , we observe:

$$0 \leq \frac{1}{a} [F_{\lambda}] \cdot [\omega_{\lambda}]^{n-d} \cdot (D_{\mathcal{X}'_{\lambda}} + \alpha_{\mathcal{X}'_{\lambda}}) \cdot \gamma^{d-1}_{\mathcal{X}'_{\lambda}},$$

since  $F_{\lambda}$  is effective, and  $D_{\lambda} + \alpha_{\mathcal{X}_{\lambda}}$ ,  $\omega$ , and  $\gamma$  are nef –which implies that  $D_{\mathcal{X}'_{\lambda}} + \alpha_{\mathcal{X}'_{\lambda}}$ ,  $[\omega_{\lambda}]$ , and  $\gamma_{\mathcal{X}'_{\lambda}}$  are nef as well.

Again by the projection formula, we have:

$$0 \leq \frac{1}{a} [F_{\lambda}] \cdot [\omega_{\lambda}]^{n-d} \cdot (D_{\mathcal{X}_{\lambda}'} + \alpha_{\mathcal{X}_{\lambda}'}) \cdot (\gamma_{\mathcal{X}_{\lambda}'})^{d-1}$$
$$= \frac{1}{a} [F] \cdot \alpha^{n-d} \cdot [\nu_{\lambda*} D_{\mathcal{X}_{\lambda}'} + \alpha_{\mathcal{X}'}] \cdot \gamma_{\mathcal{X}'}^{d-1}.$$

Now, since:

$$\nu_{\lambda*}D_{\mathcal{X}'_{\lambda}} = \sum_{\substack{\text{dired}\\G \subseteq \mathcal{X}'_{0}}} b_{G} \varphi_{D_{\lambda}}(v_{G})G \longrightarrow \sum_{\substack{\text{dired}\\G \subseteq \mathcal{X}'_{0}}} b_{G} \varphi_{D}(v_{G})G = D_{\mathcal{X}'}$$

it follows that:

$$\frac{1}{a}[F] \cdot \alpha^{n-d} \cdot [\nu_{\lambda*} D_{\mathcal{X}'_{\lambda}} + \alpha_{\mathcal{X}'}] \cdot \gamma_{\mathcal{X}'}^{d-1} \to \frac{1}{a}[F] \cdot \alpha^{n-d} \cdot (D_{\mathcal{X}'} + \alpha_{\mathcal{X}'}) \cdot \gamma_{\mathcal{X}'}^{d-1} \ge 0,$$

concluding the proof.

4.2. **PL Monge–Ampère operator and energy pairing.** In this section we will give the definition of the PL Monge–Ampère operator and, more genrally, the PL energy pairing. We also state a few important properties and results.

The notions of pluripotential theory for  $X^{\beth}$ , introduced in this paper, are under the synthetic formalism developed in [BJ23]. In particular, every result from Section 1 to 3 of Boucksom-Jonsson's synthetic approach holds in our case.

We will recall some of the results, for more details see [BJ22, Section 3.2] and [BJ23, Section 1].

4.2.1. Monge-Ampère measure of a PL function. Let  $\beta \in H^{1,1}(X)$  be a cohomology class of positive volume, i.e.  $V_{\beta} \doteq \int_X \beta^n > 0$ , and  $\varphi \in PL(X^{\beth})$  a PL function, we can associate to the pair  $(\beta, \varphi)$  a signed measure on  $X^{\beth}$ , called *Monge-Ampère measure*, given by the construction:

- Let  $\mathcal{X}$  be a snc test configuration such that there exists a vertideal divisor  $D \in \operatorname{VCar}(\mathcal{X})$  satisfying  $\varphi_D = \varphi$ .
- Denote  $\mathcal{X}_0 = \sum b_E E$  the decomposition in irreducible components of the central fiber, and let  $c_E$  be the constant given by

$$\frac{b_E}{V_{\beta}}\left((eta_{\mathcal{X}}+D)|_E
ight)^n.$$

• Define the signed measure as:

$$\mathrm{MA}_{\beta}(\varphi) \doteq \sum_{\substack{\mathrm{irred} \\ E \subseteq \mathcal{X}_0}} c_E \, \delta_{v_E}.$$

With the projection formula one checks that this definition does not depend on the choice of test configuration. Moreover, if  $\beta \in \text{Pos}(X)$  is a positive class and  $\varphi \in \mathcal{H}(\beta)$  the above construction get us a probability measure.

36

**Definition 4.2.1.** Let  $\alpha \in \text{Pos}(X)$  be Kähler class, and  $\mathcal{P}(X^{\beth})$  the set of Radon probability measures on  $X^{\beth}$ . We call the operator:

$$\mathcal{H}(\alpha) \ni \varphi \stackrel{\mathrm{MA}}{\mapsto} \mathrm{MA}_{\alpha}(\varphi) \in \mathcal{P}(X^{\beth}),$$

the Monge-Ampère operator. Whenever  $\alpha$  is it clear by context we write MA( $\varphi$ ) for  $MA_{\alpha}(\varphi).$ 

Using the identification of  $X^{\Box}$  with the limit of dual complexes, we can observe that if  $\varphi \in Aff_{\mathbb{Q}}(\Delta_{\mathcal{X}})$ , then  $p_{\mathcal{X}}^* \varphi \in PL(\Delta)$  is such that  $MA(p_{\mathcal{X}}^* \varphi)$  is supported on  $\Delta_{\mathcal{X}} \subseteq X^{\beth}$ , where  $p_{\mathcal{X}}$  is the retraction  $X^{\beth} \to \Delta_{\mathcal{X}}$ .

**Remark 4.2.2.** A borelian measure on a compact Hausdorff topological space K is completely determined by its values on a dense subset of  $C^{0}(K)$ , therefore the probability measure MA( $\varphi$ ) is completely determined by its values on the set PL( $X^{\perp}$ ). We will use this approach to generalize the Monge-Ampère measure (see Remark 4.4.8), and construct the mixed Monge–Ampère energy (see next section).

4.2.2. Energy pairing for PL functions. Let  $\varphi_0, \ldots, \varphi_n \in PL_{\mathbb{R}}$ , and  $\beta_0, \ldots, \beta_n \in H^{1,1}(X)$ .

**Definition 4.2.3.** Let  $\mathcal{X}$  be a test configuration dominating  $X \times \mathbb{P}^1$ , such that there exist  $D_0, \ldots, D_n \in \mathrm{VCar}(\mathcal{X})$ , with the property that for every  $i = 0, \ldots, n$  we have

$$\varphi_i = \varphi_{D_i}$$

then we define the energy pairing

$$(\beta_0,\varphi_0)\cdot(\beta_1,\varphi_1)\cdots(\beta_n,\varphi_n) \doteq (\beta_{0,\mathcal{X}}+D_0)\cdots(\beta_{n,\mathcal{X}}+D_n) \in \mathbb{R},$$
(4.2.1)

where, in the right hand side of the inequality, the intersection product is against the fundamental class of  $\mathcal{X}$ . As before, the projection formula quarantees that the energy pairing is well defined.

We also refer to the energy pairing as the energy coupling, or as mixed Monge-Ampère energy, see item (iv) of Proposition 4.2.4 below.

Clearly the pairing is a symmetric multi-linear form, which further satisfies the following properties:x

**Proposition 4.2.4.** Let  $\varphi_0, \ldots, \varphi_n \in PL_{\mathbb{R}}$ , and  $\mathcal{X}$  a test configuration such that there exist  $D_0, \ldots, D_n \in \operatorname{VCar}_{\mathbb{R}}(\mathcal{X})$  with

$$\varphi_i = \varphi_{D_i},$$

and let  $\beta_0, \ldots, \beta_n \in H^{1,1}(X)$ ,  $t \in \mathbb{Q}_{>0}$ , and  $\alpha \in Pos(X)$  then we have:

- $(i) \quad (0,1) \cdot (\beta_1,\varphi_1) \cdots (\beta_n,\varphi_n) = \beta_1 \cdots \beta_n$ (ii)  $(\beta_0,0) \cdots (\beta_n,0) = 0$
- (iii)  $(\beta_0, t \cdot \varphi_0) \cdots (\beta_n, t \cdot \varphi_n) = t(\beta_0, \varphi_0) \cdots (\beta_n, \varphi_n)$
- (iv) For  $\psi \in \operatorname{PL}_{\mathbb{R}}$ , and  $V_{\alpha} \doteq [X] \cdot \alpha^n$ , we have

$$\frac{1}{V_{\alpha}}(0,\psi) \cdot (\alpha,\varphi_0)^n = \int_{X^{\Box}} \psi \operatorname{MA}_{\alpha}(\varphi_0)$$

(v) More generally, for  $\psi \in PL_{\mathbb{R}}$  we have

$$(0,\psi)\cdot(\beta_1,\varphi_1)\cdots(\beta_n,\varphi_n)=\sum_{E\subseteq\mathcal{X}_0}b_E\,\psi(v_E)(\beta_1+D_1)|_E\cdots(\beta_n+D_n)|_E$$

for E irreducible component and  $b_E = v_E(\mathcal{X}_0)$ .

*Proof.* (i) Follows from the remark that, as cohomology classes,  $[\mathcal{X}_0] = [\mathcal{X}_1]$ . Indeed, both of them can be written as  $\pi^*([0])$  and  $\pi^*([1])$  respectively, where [0] and [1] represent the cohomology classes of  $0, 1 \in \mathbb{P}^1$  in  $H^2(\mathbb{P}^1, \mathbb{C}) \cong \mathbb{C}$ , but since [0] = [1], the result follows from the flatness of  $\pi: \mathcal{X} \to \mathbb{P}^1$ .

(*ii*) Observe that:

$$[\mathcal{X}_0] \cdot \beta_{0,\mathcal{X}} \cdots \beta_{n,\mathcal{X}} = [X] \cdot \beta_0 \cdots \beta_n = 0.$$

(*iii*) It is enough to check that for each  $d \in \mathbb{Z}_{>0}$  we have  $(\beta_0, d \cdot \varphi_0) \cdots (\beta_n, d \cdot \varphi_n) = d(\beta_0, \varphi_0) \cdots (\beta_n, \varphi_n)$ , but then again  $d \cdot \varphi_D = \varphi_{D_d}$ , where  $D_d$  is the pullback of D under the normalized base change:

in addition if  $\alpha_0, \ldots, \alpha_{n+1} \in H^{1,1}(\mathcal{X})$ , then  $(\alpha_0)_{\mathcal{X}_d} \cdots (\alpha_{n+1})_{\mathcal{X}_d} = d \alpha_0 \cdots \alpha_{n+1}$  and the result follows.

(iv) Follows from (v).

(v) Let  $\mathcal{X}'$  be a test configuration  $\mu$ -dominating  $\mathcal{X}$  such that there exists  $G \in \mathrm{VCar}_{\mathbb{R}}(\mathcal{X}')$  with

 $\psi = \varphi_G,$ 

then we have:

$$(0,\psi) \cdot (\beta_{1},\varphi_{1}) \cdots (\beta_{n},\varphi_{n}) = G \cdot (\beta_{1,\mathcal{X}'} + \mu^{*}D_{1}) \cdots (\beta_{n,\mathcal{X}'} + \mu^{*}D_{n})$$

$$= \mu_{*}G \cdot (\beta_{1,\mathcal{X}} + D_{1}) \cdots (\beta_{n,\mathcal{X}} + D_{n})$$

$$= \sum_{E \subseteq \mathcal{X}_{0}} \operatorname{ord}_{E}(G) E \cdot (\beta_{1} + D_{1}) \cdots (\beta_{n} + D_{n})$$

$$= \sum_{E \subseteq \mathcal{X}_{0}} b_{E} \varphi_{G}(v_{E})(\beta_{1} + D_{1})|_{E} \cdots (\beta_{n} + D_{n})|_{E}$$

$$= \sum_{E \subseteq \mathcal{X}_{0}} b_{E} \psi(v_{E})(\beta_{1} + D_{1})|_{E} \cdots (\beta_{n} + D_{n})|_{E},$$

$$(4.2.3)$$

where the second equality is given by the projection formula.

**Corollary 4.2.5.** Let  $\beta_1, \ldots, \beta_n \in \text{Pos}(X)$  and for each  $i \ a \ \beta_i$ -psh function  $\varphi_i \in \text{PL} \cap \text{PSH}(\beta_i)$ . If  $\gamma \in H^{1,1}(X)$ , and  $\psi, \psi' \in \text{PL}$  are such that  $\psi \leq \psi'$ , then

$$(\gamma,\psi) \cdot (\beta_1,\varphi_1) \cdots (\beta_n,\varphi_n) \le (\gamma,\psi') \cdot (\beta_1,\varphi_1) \cdots (\beta_n,\varphi_n)$$

*Proof.* Follows directly from equation 4.2.3.

**Lemma 4.2.6** (Zariski's Lemma). Let  $\psi$  be a PL function, and, for i = 2, ..., n, let  $\varphi_i \in PL \cap PSH(\beta_i)$ . Then

$$(0,\psi)^2 \cdot (\beta_2,\varphi_2) \cdots (\beta_n,\varphi_n) \le 0. \tag{4.2.4}$$

*Proof.* Let  $\mathcal{X}$  be a test configuration that dominates  $X \times \mathbb{P}^1$  with  $D, D_2, \ldots, D_n \in$ VCar $(\mathcal{X})$  such that  $\psi = \varphi_D$  and  $\varphi_i = \varphi_{D_i}$ .

38

The energy pairing induces a bilinear form on the finite dimension vector space  $\operatorname{VCar}_{\mathbb{R}}(\mathcal{X})$  as the map:

$$(G_1, G_2) \mapsto (0, \varphi_{G_1}) \cdot (0, \varphi_{G_2}) \cdot (\beta_2, \varphi_2) \cdots (\beta_n, \varphi_n).$$

Therefore, we must prove that this bilinear form is negative semidefinite.

The strategy will be to apply Lemma C.0.1. Let  $(E_i)$  be the irreducible components of  $\mathcal{X}_0$ , and note that they form a basis of VCar. Then:

- (1) For every  $i, j, (E_i, E_j) = E_i \cdot E_j \cdot (\beta_2 + D_2) \cdots (\beta_n + D_n)$  is non-negative, since  $\beta_K + D_k$  is nef, for k = 2, ... n.
- (2) Consider  $\mathcal{X}_0$  as an element of VCar( $\mathcal{X}$ ), then  $(\mathcal{X}_0, E_i) = \mathcal{X}_0 \cdot E_i \cdot (\beta_2 + D_2) \cdots (\beta_n + D_n) = 0$ , since every component is supported on  $\mathcal{X}_0$ .

Therefore, the bilinear form is negative semi-definite, and the result follows.

This version of the Zariski Lemma that we have just proved is what allow us to use all the synthetic pluripotential theory of [BJ23], that will provide us with important a priori estimates for the mixed energy. For more details see Appendix D.

Now we will recall some notions from –the above mentioned– synthetic pluripotential theory restricting it to our case.

Recall of some synthetic facts of Boucksom–Jonsson. If we let  $\varphi_k \in PL \cap PSH(\beta_k)$  for k = 1, ..., n - 1, and denote the symbol  $(\beta_1, \varphi_1) \cdots (\beta_{n-1}, \varphi_{n-1})$  by  $\Gamma$ , we can associate a semi-norm:

$$\|\psi\|_{\Gamma} \doteq \sqrt{-(0,\psi)^2 \cdot \Gamma},$$

for  $\psi \in PL_{\mathbb{R}}$ .

**Remark 4.2.7.** Since the (positive semi-definite) quadratic form  $-(0, \psi)^2 \cdot \Gamma$  comes from a bilinear form we have an associated Cauchy-Schwarz inequality:

 $|(0,\psi_1)\cdot(0,\psi_2)\cdot\Gamma| \le \|\psi_1\|_{\Gamma}\|\psi_2\|_{\Gamma},$ 

that is the base of the synthetic estimates of [BJ23].

**Definition 4.2.8.** Let  $\varphi, \psi \in PL \cap PSH(\alpha)$ , and denote:

$$\mathbf{J}_{\alpha}(\varphi,\psi) \doteq \frac{1}{V_{\alpha}} \sum_{j=1}^{n} \frac{j}{n+1} \|\varphi - \psi\|_{(\alpha,\varphi)^{j-1} \cdot (\alpha,\psi)^{n-j}};$$
(4.2.5)

$$\mathbf{I}_{\alpha}(\varphi,\psi) \doteq \frac{1}{V_{\alpha}} \sum_{j=1}^{n} \|\varphi - \psi\|_{(\alpha,\varphi)^{j-1} \cdot (\alpha,\psi)^{n-j}}.$$
(4.2.6)

Theorem 1.33 of [BJ23] gives that these functionals define equivalent quasi-metrics. We also recall the following definition:

**Definition 4.2.9.** We define the Monge–Ampère energy as the functional  $E_{\alpha}$ :  $PL(X^{\beth}) \rightarrow \mathbb{R}$  given by the expression:

$$\mathrm{PL} \ni \varphi \mapsto \frac{1}{(n+1)V_{\alpha}} (\alpha, \varphi)^{n+1}.$$

If  $\varphi, \psi \in PL(X^{\beth})$ , the variation of  $E_{\alpha}$  is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{E}_{\alpha} \left( t\psi + (1-t)\varphi \right) |_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{(n+1)V_{\alpha}} (\alpha, t\psi + (1-t)\varphi)^{n+1} |_{t=0}$$

$$= \frac{1}{V_{\omega}} (\alpha, \varphi)^{n} \cdot (0, \psi - \varphi)$$

$$= \int_{X^{\Box}} (\psi - \varphi) \operatorname{MA}_{\alpha}(\varphi),$$
(4.2.7)

39

justifying its name.

The energy  $E_{\alpha}$  restricts to a concave functional on  $PL \cap PSH(\alpha)$ , and thus for  $\varphi, \psi \in PL \cap PSH(\alpha)$  we get that:

$$\mathbf{E}_{\alpha}(\psi) \le \mathbf{E}_{\alpha}(\varphi) + \int_{X^{\beth}} (\psi - \varphi) \operatorname{MA}_{\alpha}(\varphi), \qquad (4.2.8)$$

and the difference

$$\mathbf{E}_{\alpha}(\varphi) - \mathbf{E}_{\alpha}(\psi) + \int_{X^{\Box}} (\psi - \varphi) \operatorname{MA}_{\alpha}(\varphi)$$

coincides with  $J_{\alpha}(\varphi, \psi)$ .

4.3. Non-archimedean psh functions. We will now define one of the most important objects of study of this paper the *non-archimedean psh functions*.

Definition 4.3.1. A function

$$\psi \colon X^{\beth} \to [-\infty, +\infty[$$

is  $\alpha$ -psh if  $\psi \not\equiv -\infty$ , and there exists a decreasing net  $(\varphi_{\lambda})_{\lambda \in \Lambda} \in PL \cap PSH(\alpha)$  such that

$$\varphi_{\lambda}(v) \searrow \psi(v), \text{ for every } v \in X^{\perp}.$$
 (4.3.1)

The set of all  $\alpha$ -psh functions will be denoted  $PSH(\alpha)$ .

Like for the PL functions we have the following properties: for  $\varphi, \psi \in PSH(\alpha)$ , and  $f: X \to Y$  a finite holomorphic map:

- (1)  $f^*\varphi \in \text{PSH}(f^*\alpha),$
- (2)  $\varphi + c$ , and  $t \cdot \varphi$  are  $\alpha$ -psh for  $c \in \mathbb{R}$  and  $t \in \mathbb{Q}$ ,
- (3)  $\max\{\varphi, \psi\} \in PSH(\alpha)$ .

The next result is well known in the algebraic case, see for instance [BJ22, Corollary 4.17] or [BFJ16], and the proof in our transcendental setting goes without a change. We follow the proof of [BFJ16, Section 6.1], which is added here for completeness.

**Theorem 4.3.2.** If  $\psi \in PSH(\alpha)$ , then

$$\psi|_{X^{\mathrm{div}}} > -\infty.$$

*Proof.* Let v be a divisorial valuation, and  $\mathcal{X}$  be a test configuration such that, decomposing the central fiber in irreducible components

$$\mathcal{X}_0 = \sum_{j=0}^k b_j E_j,$$

we have  $v = v_{E_0}$ .

Consider now,  $\gamma \in \text{Pos}(\mathcal{X})$ , and  $(\varphi_{\lambda})_{\lambda} \in \text{PL} \cap \text{PSH}(\alpha)$  such that:

$$\varphi_{\lambda}(v) \searrow \psi(v)$$
, for every  $v \in X^{\beth}$ .

We may assume  $\sup \psi = 0$ , and hence that  $\sup \varphi_{\lambda} = 0 = \max\{\varphi_{\lambda}(v_{E_j})\}.$ 

If  $\mathcal{X}_{\lambda}$  is a test configuration that  $\mu_{\lambda}$ -dominates  $\mathcal{X}$ , with  $D_{\lambda} \in \operatorname{VCar}(\mathcal{X}_{\lambda})$  such that  $\varphi_{\lambda} = \varphi_{D_{\lambda}}$ , then, since  $(\mu_{\lambda})^* E_j$  is an effective divisor, and the classes

$$\alpha_{\mathcal{X}_{\lambda}} + D_{\lambda}, \ (\mu_{\lambda})^* \gamma$$
 are relatively nef w.r.t.  $\mathbb{P}^1$ 

we have:

$$0 \le (\mu_{\lambda})^* E_j \cdot (\alpha_{\mathcal{X}_i} + D_{\lambda}) \cdot (\mu_{\lambda})^* \gamma^{n-1} = E_j \cdot (\alpha_{\mathcal{X}} + (\mu_{\lambda})_* D_{\lambda}) \cdot \gamma^{n-1}, \qquad (4.3.2)$$

where the equality is given by the projection formula.

Since 
$$(\mu_{\lambda})_* D_{\lambda} = \sum_k b_k \varphi_{D_{\lambda}}(v_{E_k}) E_k$$
, rewriting the inequality (4.3.2), it follows:  
$$\sum_k b_k \varphi_{D_{\lambda}}(v_{E_k}) (E_j \cdot E_k \cdot \gamma^{n-1}) \ge -E_j \cdot \alpha_{\mathcal{X}} \cdot \gamma^{n-1}.$$

Now, if  $E_j \cap E_k \neq$ , then  $E_j \cdot E_k \cdot \gamma^{n-1} > 0$ , and thus for all j:

$$b_j(E_j \cdot E_j \cdot \gamma^{n-1}) = E_j \cdot (b_j E_j - \mathcal{X}_0) \cdot \gamma^{n-1}$$
$$= -\sum_{k \neq j} b_k(E_j \cdot E_k \cdot \gamma^{n-1}) \le -1$$

where the first equality comes from flatness of  $\pi: \mathcal{X} \to \mathbb{P}^1$ , and the last inequality comes from  $\mathcal{X}_0$  being connected with at least two irreducible components. Exactly like [BFJ16, Section 6.1], we get:

$$|\varphi_{D_{\lambda}}(v_{E_j})| \le C(\mathcal{X}, \alpha, \gamma), \tag{4.3.3}$$

for some constant C depending only on  $\mathcal{X}, \alpha$  and  $\gamma$ . Hence:

$$C \ge \lim_{i} |\varphi_{D_{\lambda}}(v_{E_j})| = |\psi(v_{E_j})|$$

concluding the proof.

Thanks to the above result, a natural topology to endow  $PSH(\alpha)$  will be the topology of pointwise convergence on divisorial valuations.

Bellow, we will also prove that:

$$\varphi \leq \psi \text{ on } X^{\operatorname{div}} \implies \varphi \leq \psi \text{ on } X^{\beth}$$

for  $\varphi \in \text{PSH}(\alpha)$  and  $\psi \colon X^{\beth} \to [-\infty, +\infty[$  a usc function. To get this result we need the following description of divisorial valuations:

**Definition 4.3.3.** Let  $\mathfrak{a} \in \mathcal{F}$  be a flag ideal, the set  $\Sigma_{\mathfrak{a}} \subseteq (X \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*} = \sigma(X^{\text{div}})$  of divisorial valuations given by the irreducible components of the exceptional divisor of the normalized blow-up  $\operatorname{Bl}_{\mathfrak{a}} X \times \mathbb{P}^1$  are called the Rees valuations associated to  $\mathfrak{a}$ .

It is clear from definition that:

(1) 
$$\Sigma_{\mathfrak{a}} = \Sigma_{\overline{\mathfrak{a}}}$$
  
(2)  $\Sigma_{\mathfrak{a}^m} = \Sigma_{\mathfrak{a}}$   
(3)  $\bigcup_{\mathfrak{a} \in \mathscr{I}_X} \Sigma_{\mathfrak{a}} = (X \times \mathbb{P}^1)^{\operatorname{div}}_{\mathbb{C}^*}$ 

The next result is a generalization of [BJ22, Lemma 2.13], and the proof follows the same general lines.

**Lemma 4.3.4.** Let  $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}$ , and  $m \in \mathbb{N}$ , then

$$\sup_{X \rightrightarrows} \left\{ \frac{1}{m} \varphi_{\mathfrak{b}} - \varphi_{\mathfrak{a}} \right\} = \max_{\Sigma_{\mathfrak{a}}} \left\{ \frac{1}{m} \varphi_{\mathfrak{b}} - \varphi_{\mathfrak{a}} \right\}$$

*Proof.* After replacing  $\mathfrak{a}$  for  $\mathfrak{a}^m$ , we can suppose that m = 1. Set

$$C \doteq \max_{\Sigma_{\mathfrak{b}}} \{\varphi_{\mathfrak{a}} - \varphi_{\mathfrak{b}}\}$$
(4.3.4)

Let  $\mathcal{X} = \operatorname{Bl}_{\mathfrak{a}} \widetilde{X \times \mathbb{P}^1}$ , and let  $\mathcal{X}_0 = \sum b_i E_i$  be the decomposition into irreducible components of the central fiber, so that  $\Sigma_{\mathfrak{a}} = \{v_{E_1}, \ldots, v_{E_k}\}$ . Then, we observe that:

- (1) The ideal  $\mathfrak{a}$  becomes invertible on  $\mathcal{X}$ .
- (2) We can then "subtract", and Equation 4.3.4 reads

$$\operatorname{ord}_{E_i}(\mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}(D)) \ge 0, \quad \text{for every } i = 1, \dots, k$$

$$(4.3.5)$$
visor  $D \in \operatorname{VCar}(\mathcal{X}).$ 

for some divisor  $D \in \operatorname{VCar}(\mathcal{X})$ .

41

 $\square$ 

- (3) The polar variety of  $\mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}(D)$  is contained in the central fiber  $\mathcal{X}_0$ , hence Equation 4.3.5 implies that the polar variety is of codimension at least 2.
- (4) Since  $\mathcal{X}$  is normal,  $\mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}(D) \subseteq \mathcal{O}_{\mathcal{X}}$  hence proving the result.

As a consequence of the previous lemma, like in [BJ22, Lemma 4.26], we conclude that:

**Proposition 4.3.5.** Let  $\psi \in PL(X^{\beth})$ , then there exists a finite subset  $\Sigma(\psi) = \Sigma \subseteq X^{\text{div}}$  such that for every  $\varphi \in PSH(\alpha)$  we have:

$$\sup_{X^{\beth}}(\varphi - \psi) = \max_{\Sigma}(\varphi - \psi).$$

**Corollary 4.3.6.** Let  $\psi \in C^0(X^{\beth}, \mathbb{R})$ , then the function

$$\mathrm{PSH}(\alpha) \to \mathbb{R}, \quad \varphi \mapsto \sup_{X^{\beth}} (\varphi - \psi)$$

is continuous.

*Proof.* Follows from the previous lemma, together with the density of  $PL(X^{\exists})$  in  $C^{0}(X^{\exists})$ .

**Theorem 4.3.7.** Let  $\varphi \in PSH(\alpha)$ , and  $\psi \colon X^{\beth} \to \mathbb{R} \cup \{-\infty\}$  a use function, we then have:

$$\varphi \leq \psi \text{ on } X^{\operatorname{div}} \iff \varphi \leq \psi \text{ on } X^{\beth}.$$

*Proof.* If  $\psi \in PL$  this is an easy consequence of Proposition 4.3.5.

Since every continuous function is a uniform limit of PL functions, the same result holds if  $\psi \in C^0(X^{\beth})$ .

Lastly, if  $\psi$  is a decreasing limit of the net  $(\psi_{\lambda})_{\lambda} \in C^{0}(X^{\beth})$ , we have that  $\varphi \leq \psi \leq \psi_{\lambda}$ on  $X^{\text{div}}$ , and hence by the previous case  $\varphi \leq \psi_{\lambda}$ , and finally this implies that  $\varphi \leq \psi$ .  $\Box$ 

**Remark 4.3.8.** Darvas, Xia, and Zhang developed on [DXZ23] a notion of transcendental non-archimedean psh metrics. They use the formalism of Ross-Witt Nystrum of test curves on the complex manifold X, and call a non-archimedean psh metric, a maximal test curve.

Their approach is more general to define non-archimedean  $\beta$ -psh metrics for a transcendetal big class  $\beta$ . Bellow, in Section 5.2, we compare the present approach with theirs. As we will see they coincide when  $\beta$  is Kähler, and the metric is of finite energy.

4.4. Extending the energy pairing. In this section we will extend the energy pairing to general psh functions. Unlike section 4.2, the synthetic approach of Boucksom-Jonsson does not cover this singular case, even though similar generalizations can be done.

Here we will follow closely Section 7 of [BJ22].

Let  $\alpha_0, \ldots, \alpha_n \in \text{Pos}(X)$ , and  $\varphi_i \in \text{PSH}(\alpha_i)$  for  $i = 0, \ldots, n$ , we define:

$$(\alpha_0, \varphi_0) \cdots (\alpha_n, \varphi_n) \doteq \inf \left\{ (\alpha_0, \psi_0) \cdots (\alpha_n, \psi_n) : \psi_i \in \mathcal{H}(\alpha_i), \psi_i \ge \varphi_i \right\}.$$
(4.4.1)

Lemma 4.4.1. The energy pairing,

$$\prod_{i=0}^{n} \operatorname{PSH}(\alpha_{i}) \to \mathbb{R} \cup \{-\infty\}$$
$$(\varphi_{0}, \dots, \varphi_{n}) \mapsto (\alpha_{0}, \varphi_{0}) \cdots (\alpha_{n}, \varphi_{n})$$

is upper semi-continuous.

42

It is clear that the energy pairing is increasing in each variable. Hence, together with the previous lemma, we conclude that the energy pairing is continuous along decreasing nets.

*Proof of Lemma* 4.4.1. The proof follows from Corollary 4.3.6. For more details see [BJ22, Theorem 7.1].

Just like in the algebraic setting, Corollary 7.11 of [BJ22], we have the following result:

**Proposition 4.4.2.** Let  $\alpha_0, \ldots, \alpha_n \in \text{Pos}(X)$ , and for  $i = 0, \ldots, n \ \varphi_i \in \text{PSH}(\alpha_i)$ , with  $\varphi_i \leq 0$ , then

$$(\alpha_0, \varphi_0) \cdots (\alpha_n, \varphi_n) \gtrsim t^{n^2} \min_i \left\{ (\alpha_i, \varphi_i)^{n+1} \right\}$$
(4.4.2)

for  $t \in \mathbb{R}$  sufficiently large in order to satisfy  $\alpha_i \leq t\alpha_i$  for every  $i, j \in \{0, \ldots, n\}$ .

*Proof.* Theorem 1.18 of [BJ23] gives the inequality for  $\varphi_i \in PL \cap PSH(\alpha_i)$ , by taking decreasing sequences we conclude. 

We now extend the Monge–Ampère energy functional to the class of  $\alpha$ -psh functions.

**Definition 4.4.3.** Let  $\alpha \in Pos(X)$ , and  $V_{\alpha} = \int_X \alpha^n$ , we define the Monge-Ampère energy functional to be

$$E_{\alpha} \colon \mathrm{PSH}(\alpha) \to \mathbb{R} \cup \{-\infty\}$$
$$\varphi \mapsto \frac{V_{\alpha}^{-1}}{n+1} (\alpha, \varphi)^{n+1}$$

we define the set of finite energy non-archimedean potentials to be the set

 $\mathcal{E}^{1}(\alpha) \doteq \{\varphi \in \mathrm{PSH}(\alpha) : \mathrm{E}_{\alpha}(\varphi) > -\infty\}$ 

when is clear by context we may ommit  $\alpha$ . Moreover,

$$C_{abs}^1 \doteq \bigcup_{\alpha \in \operatorname{Pos}(X)} \mathcal{E}^1(\alpha)$$

As a direct consequence of Proposition 4.4.2, we have:

**Proposition 4.4.4.** Let 
$$\alpha_0, \ldots, \alpha_n \in \text{Pos}(X)$$
, and  $\varphi_i \in \mathcal{E}^1(\alpha_i)$ , then  
 $(\alpha_0, \varphi_0) \cdots (\alpha_n, \varphi_n) \in \mathbb{R}$  (4.4.3)  
is finite.

**Remark 4.4.5.** This allow us to extend the  $J_{\alpha}$  and the  $I_{\alpha}$  functionals, defined in Section 4.2, to  $\mathcal{E}^1(\alpha)$  by the formulas of Equations (4.2.5) and (4.2.6) respectively.

Moreover, the quasi-triangular inequality, and quasi-symmetry of  $J_{\alpha}$  for  $PL \cap PSH(\alpha)$ functions, pass through, taking decreasing limits, to  $\mathcal{E}^{1}(\alpha)$ .

We can also see that the pairing of Proposition 4.4.4 is additive on the forms, and hence can be extended by linearity to  $H^{1,1}(X)$ .

Indeed, fix  $\alpha_1, \ldots, \alpha_n \in Pos(X)$  and  $\varphi_i \in \mathcal{E}^1(\alpha_i)$ , for  $i = 1, \ldots, n$ , finally denote  $\Gamma \doteq (\alpha_1, \varphi_1) \cdots (\alpha_n, \varphi_n)$ , we then define:

**Definition 4.4.6.** Let  $\beta \in H^{1,1}(X)$ ,  $\varphi \in \mathcal{E}^1_{abs}$ , and  $\alpha_0, \tilde{\alpha}_0 \in \text{Pos}(X)$ , such that  $\beta = \alpha_0 - \tilde{\alpha}_0$ 

We define the energy pairing  $(\beta, \varphi) \cdot (\alpha_1, \varphi_1) \cdots (\alpha_n, \varphi_n)$  by the formula:

$$(\beta,\varphi)\cdot(\alpha_1,\varphi_1)\cdots(\alpha_n,\varphi_n) \doteq (\alpha_0+\alpha,\varphi)\cdot\Gamma - (\tilde{\alpha}_0+\alpha,0)\cdot\Gamma$$
(4.4.4)

where  $\varphi \in \mathcal{E}^1(\alpha) \subseteq \mathcal{E}^1(\alpha + \alpha_0) \subseteq \mathcal{E}^1_{abs}$ , for some  $\alpha \in \operatorname{Pos}(X)$ .

Similarly, we get the energy pairing defined on  $\prod_{k=0}^{n} (H^{1,1}(X) \times \mathcal{E}^{1}_{abs})$ .

In particular, we define another functional whose importance will become apparent in Section 6, which will be a twisted version of the Monge–Ampère energy.

**Definition 4.4.7.** Let  $\alpha \in \text{Pos}(X)$ ,  $V_{\alpha} = \int_X \alpha^n$ , and  $\beta \in H^{1,1}(X)$ , we define the Monge-Ampère twisted energy to be the functional

$$\begin{split} \mathbf{E}^{\beta}_{\alpha} \colon \mathcal{E}^{1}(\alpha) \to \mathbb{R} \\ \varphi \mapsto V_{\alpha}^{-1}(0,\beta) \cdot (\alpha,\varphi)^{n}. \end{split}$$

**Remark 4.4.8.** We can also extend the Monge–Ampère operator to the set  $\mathcal{E}^{1}(\alpha)$ . We associate to  $\varphi \in \mathcal{E}^{1}(\alpha)$  the probability measure  $MA_{\alpha}(\varphi)$  satisfying:

$$\operatorname{PL}(X^{\beth}) \ni \psi \mapsto \int_{X^{\beth}} \psi \operatorname{MA}_{\alpha}(\varphi) \doteq \frac{1}{V_{\alpha}} (0, \psi) \cdot (\alpha, \varphi)^{n}.$$

5. FROM COMPLEX TO NON-ARCHIMEDEAN GEOMETRY

In this section X will be a compact Kähler manifold, and we will fix a Kähler metric  $\omega \in \mathcal{K}(X)$ , and  $\alpha \in H^{1,1}(X)$  its cohomology class. We also will only consider smooth test configurations dominating  $X \times \mathbb{P}^1$ , with snc central fiber.

**Basic Kähler Geometry tools.** Let *I* be a coherent ideal of *X*, and  $\beta \in H^{1,1}(X)$ , we say that  $I \otimes \beta$  is *nef*, if  $-F + \beta$  is nef, for  $F \subseteq Y$  a log resolution of *I*, and *F* the effective divisor induced by *I*.

**Proposition 5.0.1.** Let I be an ideal such that there exists  $U: X \to \mathbb{R} \cup \{-\infty\}$  a  $\omega$ -psh function with of singularity type I, then  $\alpha \otimes I$  is nef.

*Proof.* Let  $\mu: Y \to X$  be a log resolution of I, and  $F \subseteq Y$  the effective divisor induced by I. We have, by Siu's decomposition theorem, that:

$$0 \le \mu^* \omega + \mathrm{dd}^{\mathrm{c}}(U \circ \mu) = \delta_F + T \tag{5.0.1}$$

for T a positive current of bounded potential, that is,

$$T = -\eta_F + \mu^* \omega + \mathrm{dd}^{\mathrm{c}} \psi$$

for  $\psi \in L^{\infty}$ , and  $\eta_F$  a smooth representative of  $c_1(\mathcal{O}(F))$ . Hence

$$\psi \in \mathrm{PSH}(-\eta_F + \mu^* \omega) \cap L^\infty$$

By a classical result due to Demailly  $[-\eta_F + \mu^* \omega]$  is nef.

**Lemma 5.0.2.** Let  $D, E \subseteq X$  effective irreducible divisors and  $\beta \in H^{1,1}(X)$ , such that  $\beta - D$  and  $\beta - E$  admit a smooth representative which is semi-positive. Then  $\beta \otimes \{\mathcal{O}(-D) + \mathcal{O}(-E)\}$  is nef.

*Proof.* Let  $h_D$  ( $h_E$  resp.) be a smooth metric on  $\mathcal{O}_X(D)$  ( $\mathcal{O}_X(E)$  resp.) such that the associated curvature  $\theta_D$  ( $\theta_E$  resp.) is a smooth form with  $\eta - \theta_D$  ( $\eta - \theta_E$  resp.) semipositive, for  $\eta$  a smooth representative of  $\beta$ .

Let  $s_D$  be the canonic section of  $\mathcal{O}_X(D)$ , and  $s_E$  of  $\mathcal{O}_X(E)$ , then  $\psi_D \doteq \log |s_D|_{h_d}$  and  $\psi_E \doteq \log |s_E|_{h_E}$  are such that:

$$\mathrm{dd}^{\mathrm{c}}\psi_D + \theta_D = [D] \ge 0$$
, and  $\mathrm{dd}^{\mathrm{c}}\psi_E + \theta_E = [E] \ge 0$ ,

and thus  $\theta_D$  ( $\theta_E$  resp.)-psh functions.

Now, since  $\eta - \theta_D$  ( $\eta - \theta_E$  resp.) is semi-positive, both  $\psi_D$  and  $\psi_E$  are  $\eta$ -psh. In particular,  $\psi \doteq \max\{\psi_D, \psi_E\}$  is  $\eta$ -psh, and has the singularity type of  $\mathcal{O}(-D) + \mathcal{O}(-E)$ , which by Proposition 5.0.1 implies that  $\beta \otimes \{\mathcal{O}(-D) + \mathcal{O}(-E)\}$  is nef.  $\Box$ 

44

5.1. Geodesic rays and non-archimedean psh functions. The goal of this section is to get the analogues of Theorem 6.2 and Theorem 6.6 from [BBJ21] in our transcendental setting. These results are essential for the non-archimedean approach for the YTD conjecture developed by Berman–Boucksom–Jonsson, of which [Li22] and the present paper rely on.

Remember that we have fixed a Kähler form  $\omega \in \mathcal{K}(X)$ , and its cohomology class  $\alpha = [\omega]$ .

5.1.1. Quick recall on geodesic rays. In this section we will use the conventions of [BBJ21].

We define a *psh ray* as a map  $U \colon \mathbb{R}_{\geq 0} \to \mathrm{PSH}(\omega)$  such that the associated S<sup>1</sup>-invariant function,

$$U: X \times \mathbb{D}^* \to [-\infty, +\infty[, \qquad U(x,\tau) \doteq U_{-\log|\tau|}, \tag{5.1.1}$$

is  $p_1^* \omega$ -psh.

Whenever a psh ray has image in  $\mathcal{E}^1(\omega)$  and  $t \mapsto E_{\omega}(U_t)$  is affine, we say that U is a psh geodesic ray.

Moreover, a psh ray U has *linear growth*, if there exist C, D > 0 such that:

$$U_t \le C t + D.$$

Every psh geodesic has linear growth, cf. [BBJ21, Proposition 4.1].

**Remark 5.1.1.** Darvas proves in [Dar17, Theorem 2] that psh geodesic rays are -a distinguished class of -a ctual geodesic rays for the Darvas metric  $d_1$ , and in [Dar15] that for  $U_0$  and  $U_1$  finite energy potentials, there always exists a psh geodesic joining them.

We will study now the relationship between –archimedean– rays of functions on X, with non-archimedean functions on  $X^{\beth}$ .

**Definition 5.1.2.** A S<sup>1</sup>-invariant function,  $U: X \times \mathbb{D}^* \to \mathbb{R} \cup \{-\infty\}$ , is  $C^{\infty}$  (resp.  $L^{\infty}$ )-compatible with  $D \in VCar(\mathcal{X})$ , for  $\mathcal{X}$  a  $\mu$ -dominating test configuration, if:

 $U \circ \mu + \log|f_D|$  locally extends to a smooth (resp. bounded) function (across  $\mathcal{X}_0$ ),

for  $f_D$  a local equation of D.

Furthermore, if U is a compatible (either smoothly, or boundedly) with the vertical divisor D we write:

$$U^{\beth} \doteq \varphi_D. \tag{5.1.2}$$

Using this new terminology, we adapt Proposition 5.0.1 to this language.

**Lemma 5.1.3.** Let U be a  $\omega$ -psh ray  $L^{\infty}$ -compatible with a vertical divisor  $D \in VCar(\mathcal{X})$ , then  $U^{\beth} = \varphi_D$  is  $\alpha$ -psh.

*Proof.* We first observe that we can suppose D effective, otherwise consider

$$\varphi_{D+c\mathcal{X}_0} = \varphi_D + c$$

for  $c \gg 0$ , that is  $\alpha$ -psh iff  $\varphi_D$  is. Let  $\mu \colon \mathcal{X} \to X \times \mathbb{P}^1$  be a morphism of test configurations, then by Siu's decomposition formula we have:

$$0 \le \omega_{\mathcal{X}} + \mathrm{dd}^{\mathrm{c}}(U \circ \mu) = -\delta_D + T,$$

for T a positive current of bounded potential, that is

$$0 \le T = \eta_D + \omega_{\mathcal{X}} + \mathrm{dd}^{\mathrm{c}}\psi$$

with  $\psi \in L^{\infty}$ , and  $\eta$  a smooth representative of  $c_1(\mathcal{O}_{\mathcal{X}}(D))$ .

Consider the irreducible decomposition  $\mathcal{X}_0 = \sum b_k E_k$ , since T is of bounded potential, we can restrict T to a bounded positive current supported on  $E_k$ :

$$0 \leq T|_{E_k}$$

And hence by a result of Demailly we have:

$$T|_{E_k}] = [T]|_{E_k} = (D + [\omega_{\mathcal{X}}])|_{E_k} \text{ is nef}$$

Therefore  $U^{\beth}$  is  $\alpha$ -psh.

More generally, for any  $U: \mathbb{R}_{>0} \to PSH(\omega)$  psh ray of linear growth there is an induced "non-archimedean" map:

$$U^{\mathrm{NA}} \colon X^{\mathrm{div}} \to \mathbb{R}$$

given by the following procedure:

- (1) Let  $E \subseteq \mathcal{X} \xrightarrow{\mu} X \times \mathbb{P}^1$  be a prime vertical divisor.
- (2) Consider the function  $V \doteq U \circ \mu \colon \mu^{-1}(X \times \mathbb{D}^*) \to [-\infty, +\infty]$ , where U is like in Equation 5.1.1.
- (3) Define

$$U^{\rm NA}(v_E) \doteq -\nu(V, E)$$

where  $\nu$  denotes the generic Lelong number along E.

The goal of the next result is to extend the above construction of  $U^{\rm NA}$  to an  $\alpha$ -psh function on  $X^{\Box}$ , generalizing Lemma 5.1.3 for a more general singularity type. This result is an analogue of Theorem 6.2 of [BBJ21], the proof here follows the same strategy as in [BBJ21] but we use directly a regularization result of Demailly, [Dem92, Proposition 3.7], without passing by the Castelnuovo-Mumford criterion of global generation (remember that in the projective case  $\alpha = c_1(L)$ .

**Theorem 5.1.4.** Let  $U: \mathbb{R}_{>0} \to PSH(\omega)$  be a psh ray of linear growth, then

$$U^{\mathrm{NA}} \colon X^{\mathrm{div}} \to \mathbb{R}$$

extends to a  $\alpha$ -psh function

$$U^{\beth} \colon X^{\beth} \to \mathbb{R} \cup \{-\infty\}.$$

**Remark 5.1.5.** By Theorem 4.3.7 if such an extension exists it is unique.

Proof of Theorem 5.1.4. We will show that there exists a sequence  $\varphi_m \in \mathcal{H}(\alpha)$  such that:

- (1)  $(\varphi_m)_m$  is decreasing (2)  $\varphi_m(v_E) \searrow U^{\text{NA}}(v_E)$  for every  $v_E \in (X \times \mathbb{P}^1)^{\text{div}}_{\mathbb{C}^*}$

Hence  $U^{\beth}(v) \doteq \lim \varphi_m(v)$  will be the desired function.

By [Dem92, Proposition 3.7], there exists a sequence of  $S^1$ -invariant functions:

 $V_m \colon X \times \mathbb{D}_{1-\epsilon} \subseteq X \times \mathbb{D} \to \mathbb{R} \cup \{-\infty\},\$ 

that we can suppose  $\omega$ -psh, for  $\mathbb{D}_{1-\epsilon}$  the disk of radius  $1-\epsilon$ , having analytic singularities of type  $\mathcal{J}(mU)^{\frac{1}{m}}$ .

Therefore,  $U_m \doteq \max\{V_m, \log|\tau|\}$  is  $\omega$ -psh, and has analytic singularities of type  $(\mathfrak{a}_m)^{\frac{1}{m}}$ for  $\mathfrak{a}_m \doteq \mathcal{J}(mU) \cdot \mathcal{O}_{X \times \mathbb{P}^1} + (t^m)$ , a flag ideal.

Moreover, if we let  $\mu_m \colon \mathcal{X}_m \to X \times \mathbb{P}^1$  be the test configuration given by the normalized blow-up of  $X \times \mathbb{P}^1$  along  $\mathfrak{a}_m$ , and  $E_m \subseteq \mathcal{X}_m$  be the exceptional divisor, then the function  $U_m \circ \mu_m$ :

(1) is  $\mu_m^* \omega$ -psh;

(2) has divisorial singularities along  $\frac{1}{m}E_m$ .

Hence,  $U_m$  is a psh ray  $L^{\infty}$ -compatible with  $\frac{1}{m}E_m$ , and by Lemma 5.1.3

$$\varphi_m \doteq U_m^{\beth} = \varphi_{\mathfrak{a}_m}$$

is  $\alpha$ -psh.

The item (ii) of Proposition 3.7 of [Dem 92] gives us that the Lelong numbers of  $V_m$ along divisors over the central fiber  $X \times \{0\}$  approach the Lelong numbers of U over the same divisors, in particular, the Lelong numbers of  $U_m$  have the same property. Thus in non-archimedean terms:

$$\varphi_m|_{X^{\mathrm{div}}} \to U^{\mathrm{NA}}$$

Moreover, by the subadditivity of multiplier ideals –like in [BBJ21, Lemma 5.7]– the sequence  $(\varphi_{2^m})_m$  is decreasing, concluding the proof. 

Now, we will prove a result in the converse direction of the above theorem. For that we remember the following definition:

**Definition 5.1.6.** A psh geodesic ray U in  $PSH(\omega)$  is maximal if for every other geodesic ray V with  $U_0 \geq V_0$  and  $U^{\square} \geq V^{\square}$  we have  $U_s \geq V_s$  for every  $s \in \mathbb{R}_{>0}$ .

Maximal geodesic rays are in correspondence with the non-archimedean potentials of finite energy, as we will see in the next theorem, an analogue of [BBJ21, Theorem 6.6] in the Kähler setting.

**Theorem 5.1.7.** Let  $\varphi \in \mathcal{E}^1(\alpha)$  be a non-archimedean potential, and  $u \in \mathcal{E}^1(\omega)$  a reference metric, then there exists a unique maximal geodesic ray  $U: [0, +\infty[ \rightarrow \mathcal{E}^1(\omega) \text{ starting}]$ at u, such that:

$$U^{\Box} = \varphi$$

**Lemma 5.1.8.** Let  $\varphi \in \mathcal{H}(\alpha)$ , and  $\mathcal{X}$  a smooth dominating test configuration with  $D \in$  $\operatorname{VCar}(\mathcal{X})$  such that  $\varphi = \varphi_D$ , and

 $D + \alpha_{\mathcal{X}}$  is Kähler relatively to  $\mathbb{P}^1$ .

Then.

- (i) there exists a psh ray, starting from  $u \in \mathcal{H}(\omega)$ , which is  $C^{\infty}$ -compatible with  $(\mathcal{X}, D)$ .
- (ii) The energy use of rays like in (i), is a maximal psh geodesic and is  $L^{\infty}$ -compatible with  $(\mathcal{X}, D)$ .

*Proof.* For (i) see [SD18, Lemma 4.4]. For (ii) we observe that in the terminology of [Ber16, Proposition 2.7], a positively curved metric  $\phi$  on a test configuration  $(\mathcal{X}, \mathcal{L})$ ,  $\mathcal{L} = L_{\mathcal{X}} + D$ , it is a psh ray, being locally bounded it is equivalent to ours  $L^{\infty}$ -compatibility with  $(\mathcal{X}, D)$ , and

$$(\mathrm{dd}^{\mathrm{c}}\varphi)^{n+1} = 0$$

is equivalent –under the positivity condition– of being a psh geodesic. Translating to our language their proof follows with no change. 

**Remark 5.1.9.** Let  $\varphi \in \mathcal{H}(\alpha)$ , and U a maximal psh geodesic s.t.  $U^{\Box} = \varphi$ , like in the previous lemma. Then by [SD18, Remark 4.11] we have that:

$$\mathbf{E}_{\omega}(U_t) = \mathbf{E}_{\omega}(U_0) + t \cdot \mathbf{E}_{\alpha}(\varphi).$$

*Proof of Theorem 5.1.7.* The proof is like the one [BBJ21, Theorem 6.6]. It relies on the following observations:

- If  $u \in \mathcal{H}(\omega)$ , and  $\varphi \in \mathcal{H}(\alpha)$ , we apply Lemma 5.1.8, and get a maximal geodesic ray connecting u and  $\varphi$ .
- Now, if  $u \in \mathcal{E}^1(\omega)$ , and  $\varphi \in \mathcal{E}^1(\alpha)$ , we can take "smoothing" decreasing sequences  $\varphi_m \in \mathcal{H}$ , and  $u_m \in \mathcal{H}(\omega)$ .

- Using the first point there exists maximal geodesic ray,  $U_{m,s}$ , uniting  $u_m$  and  $\varphi_m$ .
- By maximality  $U_{m+1} \leq U_m$ , and hence there exists the limit  $\lim U_m(x)$ , which we will denote by U(x).
- Since the energy is affine on these maximal geodesic rays, by Remark 5.1.9, and decreasing on decreasing sequences, we have an uniform lower bound on the energy of  $U_{m,s}$ . This implies that  $U_s$  is of finite energy, and a psh geodesic ray.
- Moreover, from  $U \leq U_m$  we get  $U^{\square} \leq \varphi_m$ , and thus  $U^{\square} \leq \varphi$ , but also the previous estimate gives us that  $E_{\alpha}(U^{\square}) = E_{\alpha}(\varphi)$ , and therefore  $U^{\square} = \varphi$ .
- Finally, if V is a psh ray of linear growth, such that  $V_0 \leq u \leq u_m$ , and  $V^{\beth} \leq U^{\beth} = \varphi \leq \varphi_m$ , by maximality of  $U_m$  we have  $V \leq U_m$ , and thus  $V \leq U$ , getting maximality of U and concluding the proof.

5.2. Comparison with Darvas–Xia–Zhang non-archimedean metrics. On [DXZ23] and [Xia24] the authors develop a theory of non-archimedean plurisubharmonic functions attached to a compact Kähler manifold.

Their approach is to define a non-archimedean  $\alpha$ -psh function as a *test curve* on the manifold X. Test curves are the Ross–Witt Nyström transforms of maximal geodesic rays.

Following the strategy of [DXZ23, Theorem 3.17], together with Theorem 5.1.7, for  $\beta \in \text{Pos}(X)$  a Kähler class, one can gets a correspondence from their  $\beta$ -psh functions of finite energy to ours associating to every  $\mathcal{I}$ -maximal test curve,  $\psi_{\tau}$ , the "beth" of its Ross–Witt Nyström transform  $(\check{\psi}_t)$ :

$$\mathcal{R}^{1}_{\mathcal{T}}(\theta) \ni (\psi_{\tau}) \mapsto (\check{\psi})^{\beth} \in \mathcal{E}^{1}(\beta)$$

for a smooth Kähler representative  $\theta$  of  $\beta$ .

**Remark 5.2.1.** As mentioned before, their theory remains more general since they can consider the case when  $\beta$  is big. On the other hand, our theory is a direct analogue of the algebraic setting, which for instance enables us to associate a Monge–Ampère measure to a  $\beta$ -psh function.

The comparison of  $\beta$ -psh functions –without the energy assumption– is more delicate even in the algebraic case, and we refer to [DXZ23, Theorem 3.14] for more details.

5.3. Asymptotics for the mixed energy. The goal of this section is to prove Theorem 5.3.4, which will be of central importance to relate the variational cscK problem with non-archimedean geometry. We begin with an useful lemma.

**Lemma 5.3.1.** Let  $\varphi \in \mathcal{E}^1(\alpha)$  be a non-archimedean potential, and  $U_s$  the associated maximal geodesic ray. Then

$$\frac{V_{\omega}^{-1}}{s} \int_X U_s \, \omega^n \stackrel{s \to \infty}{\longrightarrow} \varphi(v_{\text{triv}}) = \int_{X^{\Box}} \varphi \, \text{MA}_{\alpha}(0).$$

*Proof.* Observe that, since  $0 \leq \sup U_s - V_{\omega}^{-1} \int_X U_s \omega^n \leq T_{\omega}$ , we are left to prove that  $\frac{\sup U_s}{s} \to \varphi(v_{\text{triv}}) = \sup \varphi$ . Let  $\varphi_m \in \mathcal{H}(\alpha)$  be a decreasing sequence converging to  $\varphi$ , and  $U_s^m$  the associated maximal geodesic rays. Let's assume for simplicity that  $U_0 = 0 = U_0^m$ , by [BBJ21, Proposition 1.10] we have that:

$$\sup U_s = \ell \cdot s, \quad \sup U_s^m = \ell_m \cdot s,$$

for some real number  $\ell \in \mathbb{R}$ .

By Theorem B of [SD18], it follows that  $\ell_m = \sup \varphi_m = \varphi_m(v_{\text{triv}})$ , hence

$$\ell = \frac{\sup U_s}{s} \swarrow \frac{\sup U_s^m}{s} = \varphi_m(v_{\rm triv}) \searrow \varphi(v_{\rm triv}),$$

concluding the proof.

We will now recall Theorem 3.6 from [BJ23] that will be useful later. Here we will state (and use) only the complex analytic version of the theorem. Keep in mind that the same result –in its appropriate formulation– holds in our non-archimedean case, as all the synthetic theory of Boucksom–Jonsson.

**Lemma 5.3.2.** Let  $\eta_0, \ldots, \eta_n$  be smooth closed (1,1)-forms, and for  $i = 0, \ldots n$  consider  $U_i, V_i \in \mathcal{E}^1(\omega)$  normalized for  $\int U_i \omega^n = 0 = \int V_i \omega^n$ , then

$$|(\eta_0, U_0) \cdots (\eta_n, U_n) - (\eta_0, V_0) \cdots (\eta_n, V_n)| \lesssim A\left(\max_i \mathcal{J}_{\omega}(U_i, V_i)^q \cdot \max_i \left\{\mathcal{J}_{\omega}(U_i) + \mathcal{T}_{\omega}\right\}^{1-q}\right),$$

for

$$q \doteq 2^{-n}, \quad A \doteq V_{\omega} \prod_{i} (1+2\|\eta_i\|_{\omega}), \quad and \quad T_{\omega} \doteq \sup_{f \in \mathrm{PSH}(\omega) \cap C^{\infty}} \left\{ \sup f - V_{\omega}^{-1} \int f \, \omega^n \right\}.$$

**Remark 5.3.3.** The quantity  $T_{\omega}$  is well known to be finite, see for instance [BJ23, Theorem 1.26].

*Proof of Lemma 5.3.2.* Whenever  $U^i, V^i$  are smooth functions the result follows from [BJ23, Theorem 3.6].

In the general case, it suffices to take decreasing sequences of smooth potentials converging to  $U^i$  and  $V^i$  respectively, and to observe that the bound proved for smooth functions is uniform. Thus, since the energy pairing is continuous along decreasing sequences, taking limits on both sides of the inequality we conclude.

The following statement is a generalization to singular metrics of [SD18, Theorem B], and of [DR17b, Theorem 4.15] -that after a small modification can be adapted to general pairings-, and to more general functionals of [Li22, Theorem 4.1] –where they only consider the twisted Monge–Ampère energy estimate. It is the key ingredient to relate the non-archimedean pluripotential theory with the complex analytic one, which will be essential to prove Theorem 6.3.3.

For the next theorem, let  $\eta_0, \ldots, \eta_k$  be smooth closed forms.

**Theorem 5.3.4** (Slope Formula). Let  $\varphi_0, \ldots, \varphi_k \in \text{PL}$ , and  $\varphi_{k+1}, \ldots, \varphi_n \in \mathcal{E}^1(\alpha)$ . Denoting by  $U_i$  a smooth ray  $C^{\infty}$ -compatible with  $\varphi_i$  if  $i \leq k$ , and a maximal geodesic ray compatible with  $\varphi_i$  if i > k, we have:

$$\frac{1}{s}(\eta_0, U_{0,s}) \cdots (\eta_n, U_{n,s}) \stackrel{s \to \infty}{\longrightarrow} ([\eta_0], \varphi_0) \cdots ([\eta_n], \varphi_n).$$

*Proof.* When k = n the result corresponds to [SD18, Theorem B]. We restrict ourselves to the case when k = n - 1 the general case, when  $k \le n - 1$ , will be similar.

Moreover, we observe that we can suppose that  $\eta_n = \omega$ , otherwise

$$(\eta_n, U_{n,s}) \cdot \Gamma_s = (\omega, U_{n,s}) \cdot \Gamma_s - (\omega - \eta_n, 0) \cdot \Gamma_s$$

49

where  $\Gamma_s \doteq (\omega_0, U_{0,s}) \cdots (\omega_{n-1}, U_{n-1,s})$ . Therefore, denoting  $[\Gamma] \doteq ([\eta_0], \varphi_0) \cdots ([\eta_{n-1}], \varphi_{n-1})$ , and applying the result for  $\eta_n = \omega$ , we have:

$$\frac{1}{s}(\eta_n, U_{n,s}) \cdot \Gamma_s \to (\alpha, \varphi_n) \cdot [\Gamma] - ([\omega - \eta], 0) \cdot [\Gamma]$$
$$= ([\eta_n], \varphi_n) \cdot [\Gamma],$$

which, by symmetry, implies the result.

Now, let's prove the result. Let  $\varphi_n^m \searrow \varphi_n$  be a decreasing sequence of functions in  $\mathcal{H}(\alpha)$ , and  $U_n^m$  the associated maximal geodesic ray.

By [SD18, Theorem B]:

$$\frac{1}{s}(\eta_0, U_{0,s}) \cdots (\eta_{n-1}, U_{n-1,s}) \cdot (\omega, U_{n,s}^m) \xrightarrow{s \to \infty} ([\eta_0], \varphi_0) \cdots ([\eta_{n-1}], \varphi_{n-1})(\alpha, \varphi_n^m), \quad (5.3.1)$$

and as  $m \to \infty$  the right hand side converges to  $([\eta_0], \varphi_0) \cdots (\alpha, \varphi_n)$ . Thus, to complete the proof we need to check that

$$\lim_{n \to \infty} \lim_{s \to \infty} \frac{1}{s} (\eta_0, U_{0,s}) \cdots (\omega, U_{n,s}^m) = \lim_{s \to \infty} \frac{1}{s} (\eta_0, U_{0,s}) \cdots (\omega, U_{n,s}).$$

What we will do next is then to study the difference:

$$(\star)_s \doteq |(\eta_0, U_{0,s}) \cdots (\omega, U_{n,s}) - (\eta_0, U_{0,s}) \cdots (\omega, U_{n,s}^m)| = |(\eta_0, U_{0,s}) \cdots (0, U_{n,s} - U_{n,s}^m)|.$$

Denoting by  $a_{i,s}$  and  $a_{n,s}^m$  the averages  $\int_X U_{i,s} \omega^n$  and  $\int_X U_{i,s}^m \omega^n$  respectively, and by  $V_{i,s}$  and  $V_{n,s}^m$  the normalized potentials

$$U_{i,s} - a_{i,s}, \qquad U_{n,s}^m - a_{n,s}^m,$$

we observe that:

$$(\star)_s = |(\eta_0, V_{0,s}) \cdots (\eta_{n-1}, V_{n-1,s}) \cdot (0, U_{n,s} - U_{n,s}^m)|$$

and by the triangle inequality it follows that:

$$(\star)_s \le |(\eta_0, V_{0,s}) \cdots (0, V_{n,s} - V_{n,s}^m)| + |(\eta_0, V_{0,s}) \cdots (0, a_{n,s}^m - a_{n,s})| \\ = |(\eta_0, V_{0,s}) \cdots (0, V_{n,s} - V_{n,s}^m)| + a_{n,s}^m - a_{n,s}.$$

By the Lemma 5.3.1 we have that taking the slope at infinity and letting m tend to infinity the second term vanishes. Next we will focus our attention on the first term.

Note that we can suppose that, for  $i \leq n-1$ ,  $\varphi_i$  is in  $\mathcal{H}(\alpha)$  and  $U_{i,s}$  is a smooth psh ray  $C^{\infty}$ -compatible with  $\varphi_i$ . If it is not the case we write  $\varphi_i = \psi'_i - \psi''_i$  the difference of  $\mathcal{H}(\alpha)$  functions, then we consider  $U'_i$  and  $U''_i$  psh-rays that are smoothly compatible with  $\psi'_i$  and  $\psi''_i$  respectively, and the difference

$$U_{i,s} \doteq U'_s - U''_s$$

will be smoothly compatible with  $\varphi_i$ , and the result follows from linearity of the pairing.

Consequently, it follows, by Lemma 5.3.2, that, for 
$$q = 2^{-m}$$
:  
 $|(\eta_0, V_{0,s}) \cdots (\omega, V_{n,s}) - (\eta_0, V_{0,s}) \cdots (\omega, V_{n,s}^m)| \lesssim J_{\omega}(V_{n,s}, V_{n,s}^m)^q \cdot \max_i \{J_{\omega}(V_{i,s}) + T_{\omega}\}^{1-q}$ 

$$= \mathcal{J}_{\omega}(U_{n,s}, U_{n,s}^m)^q \cdot \max_i \left\{ \mathcal{J}_{\omega}(U_{i,s}) + T_{\omega} \right\}^{1-q}$$
$$\lesssim d_1(U_{n,s}, U_{n,s}^m)^q \cdot (s+T_{\omega})^{1-q},$$

where the equality follows from the constant invariance of the J functional, and the last inequality by linear growth of  $U_i$ , since it implies that  $J(U_{i,s}) \leq s$ .

Moreover, we observe that:

By maximality  $U_{n,s} \leq U_{n,s}^m$ , and

$$d_{1}(U_{n,s}, U_{n,s}^{m}) = \mathcal{E}_{\omega}(U_{n,s}^{m}) - \mathcal{E}_{\omega}(U_{n,s})$$
  
=  $(\mathcal{E}_{\omega}(U_{n,1}^{m}) - \mathcal{E}_{\omega}(U_{n,0}^{m})) s - (\mathcal{E}_{\omega}(U_{n,1}) - \mathcal{E}_{\omega}(U_{n,0})) s + C_{m}$   
=  $(\mathcal{E}_{\omega}(U_{n,1}^{m}) - \mathcal{E}_{\omega}(U_{n,1})) s - (\mathcal{E}_{\omega}(U_{n,0}^{m}) - \mathcal{E}_{\omega}(U_{n,0})) s + C_{m},$ 

for  $C_m \doteq \mathbf{E}_{\omega}(U_{n,0}^m) - \mathbf{E}_{\omega}(U_{n,0})$ .

Therefore, taking the slope at infinity, and letting m tend to infinity we have the desired result.

We have already seen the non-archimedean version of some classical functionals arising from pluripotential theory. We recall their archimedean –original– version. If  $\omega \in \mathcal{K}(X)$ is a Kähler form, and  $\eta$  is any closed (1, 1)-form we have for  $u \in \mathcal{E}^1(\omega)$ :

$$\begin{aligned} \mathbf{E}_{\omega}(u) &\doteq \frac{1}{n+1} V_{\omega}^{-1}(\omega, u)^{n+1} \\ \mathbf{E}_{\omega}^{\eta}(u) &\doteq V_{\omega}^{-1}(\eta, 0) \cdot (\omega, u)^{n} \\ \mathbf{J}_{\omega}(u) &\doteq V_{\omega}^{-1}(\omega, u) \cdot (\omega, 0)^{n} - \mathbf{E}_{\omega}(u) \end{aligned}$$

By Theorem 5.3.4, we can relate the above functionals with their non-archimedean counterpart. If  $\varphi \in \mathcal{E}^1(\alpha)$ , and U the associated maximal geodesic ray, we have:

$$\lim_{s \to \infty} \frac{\mathcal{E}_{\omega}(U_s)}{s} = \mathcal{E}_{\alpha}(\varphi), \quad \lim_{s \to \infty} \frac{\mathcal{E}_{\omega}^{\eta}(U_s)}{s} = \mathcal{E}_{\alpha}^{\beta}(\varphi), \text{ and } \quad \lim_{s \to \infty} \frac{\mathcal{J}_{\omega}(U_s)}{s} = \mathcal{J}_{\alpha}(\varphi),$$
for  $\beta = [\eta].$ 

### 6. CSCK METRICS AND THE YAU-TIAN-DONALDSON CONJECTURE

In this section we'll generalize a result by Chi Li, on the existence of cscK metrics. Let  $(X, \omega)$  again be a compact Kähler manifold, and  $\alpha = [\omega]$  the cohomology class of  $\omega$ ,  $\eta \doteq -\operatorname{Ric}(\omega)$  minus the Ricci form of  $\omega$ ,  $\zeta$  its cohomology class, and <u>s</u> the cohomological constant  $\int_X \operatorname{Ric}(\omega) \wedge \omega^{n-1}$ .

6.1. The variational approach to the cscK problem. The cscK equation is the Euler–Langrange equation for the *Mabuchi functional*:

$$M_{\omega} = \underline{s} E_{\omega} + E_{\omega}^{\eta} + H_{\mu}, \qquad (6.1.1)$$

where  $\mu$  is the probability measure associated to  $\omega^n$ , and  $H_{\mu}(u)$  is the entropy of the Monge–Ampère measure of u with respect to  $\mu$ .

By the work of Chen–Cheng, [CC21a, CC21b], there exist a unique cscK metric in  $\alpha$  if, and only if, the Mabuchi functional is coercive, that is:

$$M_{\omega} \ge \delta J_{\omega} - C$$

for some  $\delta, C > 0$ .

What we do next is to define the non-archimedean counterpart of the Mabuchi energy,  $M_{\alpha}$ , and prove that the coercivity of  $M_{\omega}$  follows from the –non-archimedean– *coercivity* over  $\mathcal{E}^1$  of  $M_{\alpha}$ .

Before studying the non-archimedean version of the entropy functional, we recall a Legendre transform formula for the archimedean entropy, if  $u \in \mathcal{E}^1(\omega)$ :

$$\mathcal{H}_{\mu}(u) = \sup_{f \in \mathcal{C}^{0}(X)} \left\{ \int_{X} f \, \mathcal{M}\mathcal{A}_{\omega}(u) - \log \int_{X} \exp(f) \, \mathrm{d}\mu \right\}.$$
(6.1.2)

6.2. Non-archimedean entropy and the non-archimedean Mabuchi functional. **Definition 6.2.1.** Let  $H_{\alpha} : \mathcal{E}^{1}(\alpha) \to \mathbb{R}$  be defined as follows

$$\mathbf{H}_{\alpha}(\varphi) \doteq \int_{X^{\Box}} A_X \, \mathrm{MA}_{\alpha}(\varphi) \tag{6.2.1}$$

where  $A_X \colon X^{\beth} \to [0, +\infty]$  is the log discrepancy function on  $X^{\beth}$ . We call  $H_{\alpha}$  the nonarchimedean entropy functional.

Moreover, in analogy to the Chen-Tian formula of Equation (6.1.1), we define the non-archimedean Mabuchi functional,  $M_{\alpha} \colon \mathcal{E}^{1}(\alpha) \to \mathbb{R}, as:$ 

$$\mathbf{M}_{\alpha} \doteq \underline{s} \, \mathbf{E}_{\alpha} + \mathbf{E}_{\alpha}^{\zeta} + \mathbf{H}_{\alpha}.$$

Let  $\mathcal{X}$  be a snc test configuration and  $\varphi \in \mathcal{E}^1(\alpha)$ , we denote by  $H^{\mathcal{X}}_{\alpha}(\varphi)$  the integral:

$$\int_{X^{\beth}} (A_X \circ p_{\mathcal{X}}) \operatorname{MA}_{\alpha}(\varphi),$$

with  $p_{\mathcal{X}}$  just like in section 3.1.

**Proposition 6.2.2.** Let  $\psi \in \mathcal{E}^1(\alpha)$ , consider  $V_s \in \mathcal{E}^1(\omega)$  the maximal geodesic ray associated, then we have

$$H_{\alpha}(\psi) \le \lim_{s \to +\infty} \frac{H_{\mu}(V_s)}{s}.$$
(6.2.2)

*Proof.* Let  $\psi \in \mathcal{E}^1(\alpha)$ , and consider  $\mathcal{X}$  a snc test configuration. As seen before  $A_X \circ p_{\mathcal{X}}$  is a PL function, let's denote it  $\varphi$ . We can write  $A_{\alpha}^{\mathcal{X}}$  in terms of  $\varphi$ :

$$H_{\alpha}^{\mathcal{X}}(\psi) = \int_{X^{\beth}} (A \circ p_{\mathcal{X}}) \operatorname{MA}_{\alpha}(\psi) = \int_{X^{\beth}} \varphi \operatorname{MA}_{\alpha}(\psi)$$
$$= (0, \varphi) \cdot (\alpha, \psi)^{n}.$$

Then, by Theorem 5.3.4:

$$(0,\varphi) \cdot (\alpha,\psi)^n = \lim_{s \to +\infty} \frac{1}{s} (0,U_s) \cdot (\alpha,V_s)^n, \qquad (6.2.3)$$

for U a smoothly compatible ray with  $\varphi$ .

On the other hand, for  $f = U_s$ :

$$\frac{1}{s}H_{\mu}(V_s) \ge \frac{1}{s} \left\{ \int_X f\omega_{V_s}^n - \log \int_X \exp(f) \,\mathrm{d}\mu \right\}$$
$$= \frac{1}{s}(0, U_s) \cdot (\omega, V_s)^n - \frac{1}{s} \log \int_X \exp(U_s) \,\mathrm{d}\mu \longrightarrow (0, \varphi) \cdot (\alpha, \psi)^n - 0 = H_{\alpha}^{\mathcal{X}}(\psi)$$

where in the limit we make use of Lemma 3.11 of [BHJ19], to get:

$$\log \int_X \exp(U_s) \,\mathrm{d}\mu = O\left(\log(s)\right).$$

Therefore,

$$\mathbf{H}_{\alpha}(\psi) = \sup_{\mathcal{X}} H_{\alpha}^{\mathcal{X}}(\psi) \le \lim_{s \to +\infty} \frac{1}{s} \mathbf{H}_{\mu}(V_s),$$

concluding the proof.

**Corollary 6.2.3.** Let  $\varphi \in \mathcal{E}^1(\alpha)$ , and  $U_s \in \mathcal{E}^1(\omega)$  the maximal geodesic ray associated. Then.

$$M_{\alpha}(\varphi) \leq \lim_{s \to +\infty} \frac{M_{\omega}(U_s)}{s}.$$
 (6.2.4)

*Proof.* Follows from Theorem 5.3.4 together with Proposition 6.2.2.

)

# 6.3. Main theorem.

**Proposition 6.3.1** (Theorem 1.2 from [Li22]). Let  $U_s \in \mathcal{E}^1$  be a geodesic ray such that the slope

$$\lim_{s \to +\infty} \frac{\mathcal{M}_{\omega}(U_s)}{s} < +\infty,$$

then U is maximal.

*Proof.* The proof goes without change as in the projective setting.

It is based on a local integrability result for the exponential of a difference of psh functions in the same singularity class, and a clever use of Jenssen's inequality. For more details see [Li22, Theorem 1.2].  $\Box$ 

**Definition 6.3.2.** Let X be a compact Kähler manifold, and  $\alpha \in \text{Pos}(X)$  a Kähler class, then  $(X, \alpha)$  is uniformly K-stable over  $\mathcal{E}^1$  if there exists  $\delta > 0$  such that:

$$M_{\alpha}(\varphi) \ge \delta J_{\alpha}(\varphi), \quad \text{for every } \varphi \in \mathcal{E}^{1}(\alpha).$$
 (6.3.1)

Now, we will prove Theorem A, the main theorem of this paper.

**Theorem 6.3.3** (Theorem A). Let  $(X, \alpha)$  be a compact Kähler manifold that is uniformly *K*-stable over  $\mathcal{E}^1$ . Then,  $\alpha$  contains a unique cscK metric.

*Proof.* By [CC21b] the existence, and uniqueness of a cscK metric is equivalent to the coercivity of the Mabuchi functional  $M_{\omega}$ , i.e. the existence of  $C, \delta > 0$  such that

$$\mathcal{M}_{\omega} \ge \delta \mathcal{J}_{\omega} - C.$$

We will proceed by contradiction.

Suppose that  $M_{\omega}$  is not coercive, then, by [BBJ21, Li22, CC21b], we can find a geodesic ray emanating from 0,  $U_s \in \mathcal{E}^1(\omega)$ , normalized so that  $\sup U_s = 0$ , such that:

$$\lim_{s \to +\infty} \frac{1}{s} \mathcal{M}_{\omega}(U_s) \le 0.$$

By Proposition 6.3.1, U is maximal, therefore it is associated to a non-archimedean potential  $\varphi \in \mathcal{E}^1(\alpha)$ .

Corollary 6.2.3 gives:

$$0 \ge \lim_{s \to +\infty} \frac{1}{s} \mathcal{M}_{\omega}(U_s) \ge \mathcal{M}_{\alpha}(\varphi)$$

but since  $(X, \alpha)$  is uniformly K-stable over  $\mathcal{E}^1$ , there exists a  $\delta > 0$  such that:

$$M_{\alpha}(\varphi) \ge \delta J_{\alpha}(\varphi) > 0,$$

yielding a contradiction.

# Appendix A. Semi-rings and tropical algebras

### A.1. Semi-rings.

**Definition A.1.1.** A triple  $(S, +, \cdot)$  is a commutative semi-ring if the following conditions hold:

- (S, +) is a commutative monoid, with identity element denoted by  $0_S^{13}$ ;
- $(S, \cdot)$  is a commutative semi-group;
- For every  $a, b, c \in S$

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c), \quad and \quad 0_S \cdot a = 0.$$

A morphism of semi-rings is a function

$$\phi\colon (S,+,\cdot)\to (R,+,\cdot)$$

mapping  $0_S$  to  $0_R$  that satisfies

$$\phi(a+b) = \phi(a) + \phi(b), \quad and \quad \phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

Whenever S and R have a multiplicative identity we ask

$$\phi(1_S) = 1_R$$

We denote the set of morphisms from  $(S, +, \cdot)$  to  $(R, +, \cdot)$  by hom(S, R)

Now some examples

**Example A.1.2.** (1) Let  $S \doteq \mathbb{R} \cup \{+\infty\}$ , considered with min as the sum, and the usual sum, +, as the semi-ring multiplication is a semi-ring. Here S has a multiplicative identity given by:

$$0_S = +\infty, \quad and \quad 1_S = 0$$

Equivalently,  $S = (\mathbb{R} \cup \{-\infty\}, \max, +)$  is isomorphic to  $(\mathbb{R} \cup \{+\infty\}, \min, +)$ .

- (2) The subset  $([0, +\infty], \min, +)$  is also a semi-ring.
- (3) Let X be a topological space,  $(C^0(X, \mathbb{R}) \cup \{-\infty\}, \max, +)$  is a semi-ring.
- (4) Let A be a commutative ring, and denote by  $\mathscr{I}(A)$  the set of ideals of finite type of A, then together with the usual sum and multiplication of ideals,  $\mathscr{I}(A)$  is a semi-ring with neutral elements given by:

$$0_S = \{0\}, and 1_S = A$$

A semi-ring S comes equipped with a natural order relation, we say

$$a \le b$$
, if  $a = b + a$ .

Whenever S is unital we denote by  $S_+$  the set:

$$S_+ \doteq \{a \in S \mid a \ge 1_S\}$$

**Lemma A.1.3.** Let S be a semi-ring with multiplicative unit, then  $S_+$  inherits a semi-ring structure restricting the operations.

*Proof.* Let  $a, b \in S_+$ , then  $1_S = a + 1_S$  and  $1_S = b + 1_S$ , hence  $a + b + 1_S = a + (b + 1_S) = a + 1_S = 1_S$ . Moreover

$$a \cdot b + 1_S = a \cdot b + 1_S + b = (a + 1_S) \cdot b + 1_S = 1_S \cdot b + 1_S = b + 1_S = 1_S$$

**Example A.1.4.** Using the notation of Example A.1.2 we have:

 $<sup>^{13}</sup>$ A monoid is a semi-group with an identity element.

(1) 
$$\mathscr{I}(A)_{+} = \mathscr{I}(A)$$
, since for every  $I \in \mathscr{I}(A)$  we have  
 $I + A = A$ 

(2) If 
$$S = C^0(X, \mathbb{R}) \cup \{+\infty\}$$
, then

$$S_+ = \mathcal{C}^0(X, \mathbb{R}_+) \cup \{+\infty\}$$

**Definition A.1.5.** A semi-ring  $(S, +, \cdot)$  is idempotent if for every  $a \in S$ 

a + a = a

**Remark A.1.6.** Every semi-ring of Example A.1.2 is idempotent.

The order relation for idempotent semi-rings reads slightly more general,  $a \leq b$  if, and only if, we can decompose

a = b + c

for some  $c \in S$ .

A.2. Tropical spectrum and restrictions. Let S be a semi-ring, we recall that the *tropical spectrum of* S is the set

TropSpec 
$$S \doteq \hom(S, \mathbb{R} \cup \{+\infty\})$$

with the pointwise convergence topology.

**Lemma A.2.1.** Let S be a unital semi-ring, the restriction induces a map

TropSpec  $S \to \hom(S_+, [0, +\infty])$ 

*Proof.* Indeed, let  $\chi \in \text{TropSpec}$ , and  $f \in S_+$ , we then have that  $1_S + f = 1_S$ , and hence  $0 = \chi(1_S) = \chi(f + 1_S) = \min{\{\chi(f), \chi(1_S)\}} \le \chi(f)$ 

**Corollary A.2.2.** Let S be a semi-ring such that  $S = S_+$ , then

TropSpec  $S = \hom(S, [0, +\infty])$ 

**Definition A.2.3.** We define a  $\mathbb{R}$ -tropical algebra,  $\mathcal{A}$ , as a  $\mathbb{R}$ -vector space together with an operation  $\{\cdot, \cdot\}$  such that  $S = (\mathcal{A} \cup \{\infty\}, \{\cdot, \cdot\}, +)$  is a semi-ring, with

$$0_S = \infty, \quad and \quad 1_S = 0$$

satisfying

$$0 \leq f \implies f \leq \lambda f$$

for  $\lambda \geq 1$  a real number.

**Remark A.2.4.** Tropical algebras are unital, and admit multiplicative inverses.

Moreover, if  $\mathcal{A} \cup \{\infty\}$  is idempotent then every element of  $\mathcal{A}$  can be written as a difference of elements of  $\mathcal{A}_+$ . Indeed, if  $f \in \mathcal{A}$  we can write

$$f = -\{-f, 0\} - (-\{f, 0\})$$

and we have

$$\{0, -f\} = \{0, \{0, -f\}\} \implies 0 = \{-\{0, -f\}, 0\}$$
(A.2.1)

which implies  $0 \leq -\{0, -f\}$  and therefore  $-\{0, -f\} \in \mathcal{A}_+$ , we proceed similarly for  $-\{f, 0\}$ .

Example A.2.5. The two main examples are:

•  $\mathbb{R}$  is a tropical algebra.

•  $C^0(K, \mathbb{R})$  is a tropical algebra.

**Lemma A.2.6.** Let  $\mathcal{A}$  be an idempotent tropical algebra, then

TropSpec( $\mathcal{A} \cup \{\infty\}$ ) = { $\varphi \in \mathcal{A}^* \mid \varphi(\{f, g\}) = \max\{\varphi(f), \varphi(g)\}$ },

where  $\mathcal{A}^*$  denotes the algebraic dual.

*Proof.* For a max commuting linear functional  $\varphi \in \mathcal{A}^*$ , we can define  $\varphi(\infty)$  as  $+\infty$  and  $\varphi \in \operatorname{TropSpec}(\mathcal{A} \cup \{\infty\})$ .

On the other hand, if  $\chi \in \operatorname{TropSpec}(\mathcal{A} \cup \{\infty\})$ , we observe that taking  $f \in \mathcal{A}$  we have

$$0_{\mathbb{R}} = \chi(0_{\mathcal{A}}) = \chi(f + (-f)) = \chi(f) + \chi(-f)$$

getting that  $\chi$  is finite and Q-linear on  $\mathcal{A}$ .

We are left to prove that  $\chi$  is  $\mathbb{R}$ -linear. Indeed if  $\lambda \in \mathbb{R}_{>0}$ ,  $p_n \in \mathbb{Q}_{>0}$  an increasing sequence,  $q_n \in \mathbb{Q}_{>0}$  a decreasing sequence, both converging to  $\lambda$ , and  $f \in \mathcal{A}_+$ , we then have:

$$0 \le p_n f \le p_{n+1} f \le \lambda f \le q_{n+1} f \le q_n f,$$

thus getting

$$p_n\chi(f) \le \chi(\lambda f) \le q_n\chi(f),$$

taking the limit we get:

$$\chi(\lambda f) = \lambda \chi(f).$$

Applying Remark A.2.4 and the Q-linearity we get the desired result.

# A.3. **PL spaces.** Let K be a compact Hausdorff topological space.

**Definition A.3.1.** A PL structure on K is a Q-linear subspace of the set of continuous functions,  $PL(K) \subseteq C^0(K, \mathbb{R})$ , such that:

- It separates points;
- It contains all the Q-constants;
- It is stable by max.

We refer to the pair (K, PL(K)) as a PL space.

A map  $f: K_1 \to K_2$  is a morphism of PL spaces if it is continuous and

$$f^* \colon \mathrm{C}^0(K_2, \mathbb{R}) \to \mathrm{C}^0(K_1, \mathbb{R})$$

maps  $PL(K_2)$  to  $PL(K_1)$ . Moreover, it is an isomorphism of PL structures if the induced map  $f^* \colon PL(K_2) \to PL(K_1)$  is bijective.

**Remark A.3.2.** If PL(K) is a PL structure on K, then  $(PL, \max, +)$  is an idempotent semiring,  $PL_{\mathbb{R}}(K) \doteq PL(K) \otimes \mathbb{R}$  is a subtropical algebra of  $C^{0}(K, \mathbb{R})$ , and by Proposition 3.1.3

$$K \simeq (\operatorname{TropSpec} \operatorname{PL}_{\mathbb{R}}(K)) \setminus \{0\}/\mathbb{R}_{>0}.$$

In particular, an isomorphism of PL structures  $f: K_1 \to K_2$  is also a homeomorphism.

Appendix B. Monomial valuations and Lelong-Kiselman numbers

B.1. The Lelong–Kiselman number. Let  $\Delta \subseteq \mathbb{C}^n$  be the unit polydisk centered at 0,  $T \doteq (S^1)^n$  the compact torus, and  $w \in \mathbb{R}^n_+$ . Consider  $\varphi \colon \Delta^* \to \mathbb{R}_{\leq 0}$  a psh function.

**Definition B.1.1.** For  $w \in (\mathbb{R}_+)^n$  the Lelong-Kiselman number of  $\varphi$  at 0 with weight w is given by

$$\nu_w(\varphi, 0) \doteq \sup\left\{\delta > 0 \mid \exists U \text{ open neighborhood of } 0, \varphi(z) \le \delta \max_{w_i \ne 0} \frac{\log|z_i|}{w_i}, \ \forall z \in U\right\}$$

More generally, given a complex manifold X,  $p \in X$  and  $\psi: X \to \mathbb{R}_{\leq 0} \cup \{-\infty\}$  a quasi-psh function on X we can define  $\nu_w(\psi, p)$  similarly.

We will show now that if  $p \in \bigcap_{w_i \neq 0} \{z_i = 0\} \subseteq \Delta$ , then  $\nu_w(\varphi, 0) = \nu_w(\varphi, p)$ . To do that, we'll see that it is locally independent of p, more precisely we'll show that if there exists a open neighborhood U of 0, such that  $\varphi(z) \leq \nu_w(\varphi, 0) \min_{w_i \neq 0} \frac{\log |z_i|}{w_i}$  for every  $z \in U$ , then the inequality

$$\varphi(z) \le \nu_w(\varphi, 0) \max_{w_i \ne 0} \frac{\log |z_i|}{w_i}$$

holds for every  $z \in \Delta$ .

For that, define

$$\tilde{\varphi} \colon \Delta^* \to \mathbb{R}_{\leq 0}$$
$$z \mapsto \sup_{\xi \in T} \varphi(\xi \cdot z)$$

By the maximum principle  $\tilde{\varphi}(z) = \sup_{|\alpha_i| < 1} \varphi(\alpha_1 z_1, \dots, \alpha_n z_n).$ 

Now, since  $\tilde{\varphi}$  is *T*-invariant, there exists a convex function  $\chi: (\mathbb{R}_+)^n \to \mathbb{R}_{\leq 0}$ , such that

$$\tilde{\varphi}(z) = \chi\left(-\log|z_1|, \dots, -\log|z_n|\right) \tag{B.1.1}$$

where  $\chi$  is decreasing in each variable, in particular it is also decreasing on rays, i.e. for every  $w \in (\mathbb{R}_+)^n$  the map  $t \mapsto \chi(t \cdot w)$  is decreasing.

Now, since every bounded above decreasing convex function has a finite slope at infinity we can define:

$$\chi'_{\infty}(w) \doteq \lim_{t \to +\infty} \frac{\chi(tw)}{t}.$$
 (B.1.2)

It is clear that  $\chi'_{\infty}(w) = -\nu_w(\varphi, 0)$ . We have, for t > 0,  $\frac{\chi(tw) - \chi(0)}{t} \le \chi'_{\infty}(w)$ , hence

$$\chi(tw) \le t \cdot \chi'_{\infty}(w) + \chi(0) \le t \cdot \chi'_{\infty}(w)$$
(B.1.3)

For  $t = -\max_{w_i \neq 0} \frac{\log |z_i|}{w_i} = \min_{w_i \neq 0} \frac{-\log |z_i|}{w_i}$  gives us the following inequality:

$$\varphi(z) \leq \tilde{\varphi}(z) = \chi(-\log|z_1|, \dots, -\log|z_n|) \leq$$

$$\chi(tw) \leq \chi'_{\infty}(w) \min_{w_i \neq 0} \frac{-\log|z_i|}{w_i}$$

$$= \nu_w(\varphi, 0) \max_{w_i \neq 0} \frac{\log|z_i|}{w_i}$$
(B.1.4)

for every  $z \in \Delta$ .

B.2. An analytic interpretation of monomial valuations. Now, let  $B = \sum_{i \in I} B_i$ be a reduced snc divisor, Z a connected component of the intersection, and  $p \in Z \subseteq X$ . Let  $I \in \mathscr{I}_X$ , and  $f = \log |I|$  a quasi-psh function  $f: X \to \mathbb{R} \cup \{-\infty\}$ , with singularities

along I.

Then the monomial valuation defined by Equation (2.3.1) satisfies:

$$v_{w,Z,p}(I) = \nu_w(f,p).$$
 (B.2.1)

In particular, by the discussion the previous section, the right hand side does not depend on locally on  $p \in Z$ . Hence the left hand side also does not depend locally on  $p \in Z$ , therefore retrieving Proposition 2.3.4.

### Appendix C. Basic linear algebra of bilinear forms

**Lemma C.0.1.** Let V be a real finite dimensional vector space, and  $B: V \times V \to \mathbb{R}$  a symmetric bilinear form, such that

- (1) There exists a base  $e_i \in V$  such that  $B(e_i, e_j) \ge 0$
- (2) There exist an element on the kernel,  $v = \sum_i v^i e_i \in V$ , that is for every i:

$$B(v, e_i) = 0,$$

such that  $v^j > 0$  for every j. Then B is negative semi-definite.

*Proof.* After changing bases we can suppose that  $v = \sum_{i} e_i$ , and then the second item reduces to

$$B(e_i, e_i) = -\sum_{j \neq i} B(e_j, e_i)$$

and thus taking  $x = \sum_j x^j e_j \in V$ ,

$$B(x,x) = \sum_{i} (x^{i})^{2} B(e_{i}, e_{i}) + \sum_{i \neq j} x^{i} x^{j} B(e_{i}, e_{j})$$
$$= \sum_{i \neq j} (x^{i} x^{j} - (x^{i})^{2}) B(e_{i}, e_{j})$$
$$= \sum_{i \neq j} (x^{i} x^{j} - (x^{j})^{2}) B(e_{i}, e_{j})$$

where the last equality is given by symmetry. Thus, by changing the roles of i and j,

$$2B(x,x) = -\sum_{i,j} \left( (x^i)^2 + (x^j)^2 \right) B(e_i, e_j) \le 0$$

we get the desired result.

# APPENDIX D. A SYNTHETIC COMMENT

In this paper we assume the synthetic pluripotential theory developed in [BJ23], for  $X^{\beth}$ , where X is a compact Kähler manifold.

In fact,

- The set  $X^{\Box}$  is underlying the compact Hausdorff topological space.
- The "smooth" test functions  $\mathcal{D}$  are the set  $\operatorname{PL}_{\mathbb{R}}(X^{\beth}) \simeq \varinjlim_{\mathcal{X}} \operatorname{VCar}_{\mathbb{R}}(\mathcal{X})$ , which is dense in  $\operatorname{C}^{0}(X^{\beth}, \mathbb{R})$  by Proposition 1.4.4.
- The vector space  $\mathcal{Z}$  in our case corresponds to  $\lim_{\mathcal{X}} H^{1,1}(\mathcal{X}/\mathbb{P}^1)$ .
- The dd<sup>c</sup>:  $\mathcal{D} \to \mathcal{Z}$  operator assigns:

$$\operatorname{PL}_{\mathbb{R}} \ni \varphi_D \mapsto [\operatorname{c}_1(\mathcal{O}_{\mathcal{X}}(D))]$$

58

• For  $\beta \in \mathcal{Z}$ 

$$\beta \geq 0$$
,

- if β<sub>X</sub> ∈ Nef(X/P<sup>1</sup>) for some determination β<sub>X</sub>.
  The dimension of X<sup>□</sup> is defined to be dim X.
  The assignment Z<sup>n</sup> → C<sup>0</sup>(X<sup>□</sup>)<sup>∨</sup>, (β<sub>1</sub>,..., β<sub>n</sub>) ↦ β<sub>1</sub> ∧ ··· ∧ β<sub>n</sub>, is given by

 $\mathbf{C}^{0}(X^{\beth}) \ni f \mapsto \inf \left\{ (0, \varphi) \cdot (0, \beta_{1}) \cdots (0, \beta_{n}) \mid \varphi \geq f \right\},\$ 

that satisfies all the required properties by all the results on Section 4.2, in particular the version of Zariski's Lemma of Lemma 4.2.6 gives the seminegativity of r

$$\mathcal{D} \times \mathcal{D} \ni (\varphi, \psi) \mapsto \int_{X^{\Box}} \varphi \, \mathrm{dd}^{\mathrm{c}} \psi \wedge \beta_1 \wedge \cdots \wedge \beta_n,$$

for  $\beta_i \geq 0$ .

### References

- [Art86] M. Artin. Néron Models, pages 213–230. Springer New York, New York, NY, 1986.
- [BB17] Robert J. Berman and Bo Berndtsson. Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics. J. Am. Math. Soc., 30(4):1165–1196, 2017.
- [BBE<sup>+</sup>19] Robert J. Berman, Sebastien Boucksom, Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties. J. Reine Angew. Math., 751:27–89, 2019.
- [BBJ21] Robert J. Berman, Sébastien Boucksom, and Mattias Jonsson. A variational approach to the Yau-Tian-Donaldson conjecture. J. Am. Math. Soc., 34(3):605–652, 2021.
- [Ber90] Vladimir G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields, volume 33 of Math. Surv. Monogr. Providence, RI: American Mathematical Society, 1990.
- [Ber16] Robert J. Berman. K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics. *Inventiones mathematicae*, 203(3):973–1025, 2016.
- [BFJ15] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Solution to a non-Archimedean Monge-Ampère equation. J. Am. Math. Soc., 28(3):617–667, 2015.
- [BFJ16] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Singular semipositive metrics in non-Archimedean geometry. J. Algebraic Geom., 25(1):77–139, 2016.
- [BHJ17] Sébastien Boucksom, Tomoyuki Hisamoto, and Mattias Jonsson. Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs. Annales de l'Institut Fourier, 67(2):743– 841, 2017.
- [BHJ19] Sébastien Boucksom, Tomoyuki Hisamoto, and Mattias Jonsson. Uniform K-stability and asymptotics of energy functionals in Kähler geometry. J. Eur. Math. Soc. (JEMS), 21(9):2905– 2944, 2019.
- [Bin76] Jürgen Bingener. Schemata über Steinschen Algebren. Schr. Math. Inst. Univ. Münster (2), page 52, 1976.
- [BJ22] Sébastien Boucksom and Mattias Jonsson. Global pluripotential theory over a trivially valued field. Annales de la Faculté des sciences de Toulouse : Mathématiques, Ser. 6, 31(3):647–836, 2022.
- [BJ23] Sebastien Boucksom and Mattias Jonsson. Measures of finite energy in pluripotential theory: a synthetic approach, 2023. arXiv:2307.01697 [math.CV].
- [BJT24] Sébastien Boucksom, Mattias Jonsson, and Antonio Trusiani. Extremal Kähler metrics on blowups, 2024. In preparation.
- [Bou18] Sébastien Boucksom. Singularities of plurisubharmonic functions and multiplier ideals. http: //sebastien.boucksom.perso.math.cnrs.fr/notes/L2.pdf, 2018. [Online; accessed 11-June-2024].
- [CC21a] Xiuxiong Chen and Jingrui Cheng. On the constant scalar curvature Kähler metrics. I: A priori estimates. J. Am. Math. Soc., 34(4):909–936, 2021.
- [CC21b] Xiuxiong Chen and Jingrui Cheng. On the constant scalar curvature Kähler metrics. II: Existence results. J. Am. Math. Soc., 34(4):937–1009, 2021.
- [CDS15a] Xiuxiong Chen, Simon Donaldson, and Song Sun. Kähler-Einstein metrics on Fano manifolds.
   I: Approximation of metrics with cone singularities. J. Am. Math. Soc., 28(1):183–197, 2015.
- [CDS15b] Xiuxiong Chen, Simon Donaldson, and Song Sun. Kähler-Einstein metrics on Fano manifolds.
   II: Limits with cone angle less than 2π. J. Am. Math. Soc., 28(1):199–234, 2015.
- [CDS15c] Xiuxiong Chen, Simon Donaldson, and Song Sun. Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches  $2\pi$  and completion of the main proof. J. Am. Math. Soc., 28(1):235-278, 2015.
- [Che00] Xiuxiong Chen. The Space of Kähler Metrics. Journal of Differential Geometry, 56(2):189 234, 2000.
- [CSW18] Xiuxiong Chen, Song Sun, and Bing Wang. Kähler-Ricci flow, Kähler-Einstein metric, and K-stability. Geom. Topol., 22(6):3145–3173, 2018.
- [Dar15] Tamás Darvas. The Mabuchi geometry of finite energy classes. Adv. Math., 285:182–219, 2015.
- [Dar17] Tamás Darvas. Weak geodesic rays in the space of Kähler potentials and the class  $\mathcal{E}(X, \omega)$ . J. Inst. Math. Jussieu, 16(4):837–858, 2017.
- [Dem92] Jean-Pierre Demailly. Regularization of closed positive currents and intersection theory. J. Algebraic Geom., 1(3):361–409, 1992.
- [DR17a] Tamás Darvas and Yanir A. Rubinstein. Tian's properness conjectures and Finsler geometry of the space of Kähler metrics. J. Am. Math. Soc., 30(2):347–387, 2017.

61

- [DR17b] Ruadhaí Dervan and Julius Ross. K-stability for Kähler manifolds. Math. Res. Lett., 24(3):689– 739, 2017.
- [DS16] Ved Datar and Gábor Székelyhidi. Kähler-Einstein metrics along the smooth continuity method. *Geom. Funct. Anal.*, 26(4):975–1010, 2016.
- [DXZ23] Tamás Darvas, Mingchen Xia, and Kewei Zhang. A transcendental approach to non-Archimedean metrics of pseudoeffective classes, 2023. arXiv:2302.02541 [math.AG].
- [DZ24] Tamás Darvas and Kewei Zhang. Twisted Kähler-Einstein metrics in big classes, 2024. arXiv:2208.08324 [math.DG].
- [Fuj78] Akira Fujiki. Closedness of the Douady spaces of compact Kähler spaces. Publ. Res. Inst. Math. Sci., 14:1–52, 1978.
- [Ful98] William Fulton. Intersection theory., volume 2 of Ergeb. Math. Grenzgeb., 3. Folge. Berlin: Springer, 2nd ed. edition, 1998.
- [GR12] H. Grauert and R. Remmert. Coherent Analytic Sheaves. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012.
- [HS74] Reese Harvey and Bernard Shiffman. A characterization of holomorphic chains. Ann. Math. (2), 99:553–587, 1974.
- [JM12] Mattias Jonsson and Mircea Mustaţă. Valuations and asymptotic invariants for sequences of ideals. Annales de l'Institut Fourier, 62(6):2145–2209, 2012.
- [JM14] Mattias Jonsson and Mircea Mustață. An algebraic approach to the openness conjecture of demailly and kollár. Journal of the Institute of Mathematics of Jussieu, 13(1):119–144, 2014.
- [KM98] János Kollár and Shigefumi Mori. Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti, volume 134 of Camb. Tracts Math. Cambridge: Cambridge University Press, 1998.
- [Kol97] János Kollár. Singularities of pairs. In Proceedings of Symposia in Pure Mathematics, volume 62, pages 221–288. American Mathematical Society, 1997.
- [Laz17] R.K. Lazarsfeld. Positivity in Algebraic Geometry II: Positivity for Vector Bundles, and Multiplier Ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2017.
- [Li22] Chi Li. Geodesic rays and stability in the lowercase cscK problem. Ann. Sci. Éc. Norm. Supér. (4), 55(6):1529–1574, 2022.
- [MN15] Mircea Mustață and Johannes Nicaise. Weight functions on non-Archimedean analytic spaces and the Kontsevich-Soibelman skeleton. *Algebr. Geom.*, 2(3):365–404, 2015.
- [PT24] Chung-Ming Pan and Tat Dat Tô. Singular weighted cscK metrics on Kähler varieties, 2024. In preparation.
- [SD18] Zakarias Sjöström Dyrefelt. K-semistability of cscK manifolds with transcendental cohomology class. J. Geom. Anal., 28(4):2927–2960, 2018.
- [Szé16] Gábor Székelyhidi. The partial  $C^0$ -estimate along the continuity method. J. Am. Math. Soc., 29(2):537–560, 2016.
- [Thu07] Amaury Thuillier. Toroidal geometry and non-archimedean analytic geometry. Application to the homotopy type of certain formal schemes. *Manuscr. Math.*, 123(4):381–451, 2007.
- [Tia15] Gang Tian. K-stability and Kähler-Einstein metrics. Commun. Pure Appl. Math., 68(7):1085– 1156, 2015.
- [Xia24] Mingchen Xia. Operations on transcendental non-Archimedean metrics, 2024. arXiv:2312.17150 [math.AG].

SORBONNE UNIVERSITÉ AND UNIVERSITÉ PARIS CITÉ, CNRS, IMJ-PRG, F-75005 PARIS, FRANCE *Email address*: piccione@imj-prg.fr